A mixed 3D-Shell analytical model for the prediction of sound transmission through sandwich cylinders

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The sound transmission through an infinite multilayer cylinder composed of orthotropic skins and an isotropic polymer core is calculated analytically. The motions of the two thin orthotropic skins are described with the first-order shear deformation theory while the isotropic core is modeled with the three-dimensional elasticity theory. The polymer core transfer matrix relating the displacements and the stresses at the two common interfaces between the core and the skins is first calculated. The coupling of the two skins is then made using the modal transfer matrix of the core, leading to the global dynamic equilibrium of the multilayer cylinder. The sound Transmission Loss (TL) of the cylinder excited by an acoustic plane wave is finally calculated. Our results are compared with results published recently in the literature. Excellent agreement is observed for thin cores where the three layers vibrate in phase in the radial direction. The usefulness of the three-dimensional model is demonstrated for a thick and soft core in the higher frequency domain where the skins are vibrating out of phase with a relative displacement in the radial direction. Finally, a parametric study is conducted to demonstrate the influence of the damping of each layer and some observations are made on the shear and compressional strain energies of each layer.

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1. Introduction

In aeronautics and space industries, the structures are always designed to be as light as possible. However, a compromise must be found between the mechanical robustness and the acoustic efficiency of the structure. An optimization tool is then often necessary to improve the mechanical robustness while reducing the total weight of the structure and decreasing the acoustic pressure level in the aircraft cabin or in the payload compartment of the space launcher [1]. Consequently, fast and accurate analytical models dedicated to the calculation of sound transmission through cylindrical structures are of great interest in the aeronautic and space industries.

Several works have been done on this topic. Some of them have been devoted to isotropic structures, and have used the classical shell theories to model the cylinder vibrations. Among these works, Smith [2] presented a method to estimate the
sound transmission through a thin isotropic cylindrical shell. The same problem was investigated by White \[3\] with a finite cylinder. Later, Koval \[4\] extended the work of Smith \[2\] to simulate in-flight conditions with a uniform external airflow and an internal shell pressurization. The limits of the theoretical models were finally presented in the experimental work of Lee and Kim in the case of single \[5\] and double-walled \[6\] isotropic cylinders.

The three-dimensional elasticity theory has also been applied to isotropic cylinders. Sastry and Munjal \[7,8\] studied for example the sound transmission through an infinite multilayer isotropic cylinder excited by an incident plane wave. Recently, the case of a finite cylinder was also studied similarly by Hosseini-Toudeshky et al. \[9\] with fixed-end boundary conditions and several acoustic loadings, including uniform incident waves, monopole and dipole sources.

The sound transmission through single orthotropic or multilayer orthotropic cylinders such as composite cylinders was also studied in the past. Koval \[10\] studied for example the case of an orthotropic cylinder excited by an oblique plane wave. The equations of motion were given by Nelson et al. \[11\]. Koval \[12\] studied later the acoustic transmission through a laminated composite cylindrical shell. This time, the equations of motion formulated by Bert et al. \[13\] were employed. Later, Blaise et al. \[14\] proposed an extension of Koval’s work \[10\] to calculate the diffuse field transmission coefficient. They used Donnell–Mushati’s displacement field. In another study, Blaise and Lesueur \[15\] presented a model for the acoustic transmission through orthotropic multilayer cylindrical shells. The displacement field was still assumed without deformation in the thickness, and was defined by a modified Kirchhoff–Love model with transverse shearing. The same authors proposed also an improved model in which the deformations in the thickness of each layer were taken into account \[16,17\]. Later, in the context of laminated composite cylinders, Ghinet et al. \[18\] compared two models: a symmetrical laminate composite model where the displacement field was defined globally in the thickness, and a discrete thick laminate composite model where the displacement field and the equations of motion were defined for each layer. In both models, the displacement field was given by the Mindlin theory or the First-order Shear Deformation Theory (FSDT), and the effects of membrane, bending, transverse shearing, rotational inertia and orthotropic ply angles of each layer were considered. In a recent paper, Daneshjou et al. \[19\] used the same FSDT for a single orthotropic cylinder, and compared their results with those obtained with the Classical Thin Shell Theory (CST) where transverse shearing effects are neglected. They showed that transverse shearing had no influence in the low frequency domain but could lead to an overestimated Transmission Loss (TL) in high frequency. Daneshjou et al. \[20\] also developed a Third-order Shear Deformation Theory (TSDT) to calculate the Transmission Loss of relatively thick cylinders made by Functionally Graded Materials (FGM). They made a comparison between results obtained with CST, FSDT and TSDT for different geometric ratios $R/h$ where $h$ is the thickness and $R$ is the radius of the cylinder. They showed that the three methods give similar results in the low frequency domain, but differences appear at higher frequencies. Then, to study the acoustic wave transmission through laminated composite double-walled cylindrical shells lined with porous materials, Daneshjou et al. \[21\] coupled the CST for the inner and outer composite skins and an equivalent fluid model for the core porous layer. In a very recent paper, Zhou et al. \[22\] studied the same configuration as Ref. \[21\] with the effect of external mean flow. This time, Love’s thin shell theory was used to describe the motion of the skins.

Two kinds of analytical models can be distinguished in the literature review. The first kind is based on the three-dimensional elasticity theory, which is limited to isotropic cylinders. The second kind, based on the theory of shells and that can be used for orthotropic cylinders, is not adapted for thick and soft layers since it assumes the same radial displacement of all layers.

This paper presents a different approach to calculate the sound transmission through an infinite multilayer cylinder composed of two orthotropic thin skins separated by an isotropic polymer core. The First-order Shear Deformation Theory is used to describe the motion of the two orthotropic thin skins with membrane, bending, transverse shearing and rotational inertia effects. On the other hand, the three-dimensional elasticity theory is used to describe the motion of the isotropic polymer core. In contrary to existing CST, FSDT and TSDT, the proposed mixed 3D-Shell analytical approach has the great advantage of taking into account the relative radial displacements of the skins occurring at high frequency. Also, the FSDT used for the inner and outer skins remain valid at very high frequency, since the geometric ratio $R/h$ is about 1000 for the application presented in the present work. Indeed, as presented in Ref. \[20\], in the range $R/h \geq 100$ the FSDT remains very accurate when compared to the TSDT.

The first part of the paper presents the equations governing the motion of the sandwich structure. The isotropic core and the orthotropic skins are presented separately. Then the Modal Transfer Matrix Method (MTMM) is used to calculate the surface impedance matrix of the polymer core, which couples the displacement fields of the inner and outer skins. Finally, the vibroacoustic problem depicted in Fig. 1 is studied and the Transmission Loss (TL) of a sandwich cylinder excited by an oblique incident plane wave is calculated. In the second part, the results are compared with those published in the literature. Finally, numerical investigations are performed to highlight the effects of various parameters of the sandwich structure on the TL.

2. Vibrations of an infinite multilayer cylinder

This section presents the equations of motion of the skins derived from the First-order Shear Deformation Theory and the Modal Transfer Matrix Method of the core obtained with the three-dimensional linear elasticity theory. The continuity conditions and the transfer matrix are then used to couple the two skins. In the following, layers 1 and 3 refer to the inner...
and outer skins respectively, and layer 2 designates the isotropic polymer core. The notations are presented in detail in Fig. 2.

2.1. Equations of motion of the two orthotropic skins

For each skin \(i\) \((i = 1, 3)\), the displacement field is given by the First-order Shear Deformation Theory (FSDT):

\[
\begin{align*}
\mathbf{u}_i(z, \theta, \xi) &= \mathbf{u}_i^0(z, \theta) + \xi \psi^i_z(z, \theta), \\
\mathbf{v}_i(z, \theta, \xi) &= \mathbf{v}_i^0(z, \theta) + \xi \psi^i_\theta(z, \theta), \\
\mathbf{w}_i(z, \theta, \xi) &= \mathbf{w}_i^0(z, \theta),
\end{align*}
\]  

(1a)  
(1b)  
(1c)

where \(\mathbf{u}_i^0, \mathbf{v}_i^0\) and \(\mathbf{w}_i^0\) are the displacements at \(\xi = 0\) of the layer \(i\) in the axial, circumferential and radial directions, respectively, and \(\psi^i_z\) and \(\psi^i_\theta\) are the rotations of the normal to the median surface of each layer \(i\) (see Fig. 3). Note that the \(\xi\)-axis origin is at the median surface of the cylinder (see Fig. 2).
The generalized equations of motion are written for each skin in the following form (see Ref. [23] for more details):

\[
\frac{\partial N_i^i}{\partial z} + \frac{\partial N_{i\theta}^i}{\partial \theta} + q^i_z = \left( T_i^0 u_0^i + T_2^i \psi_2^i \right),
\]
(2a)

\[
\frac{\partial N_i^\theta}{\partial \theta} + \frac{\partial N_{i\theta}^\theta}{\partial \theta} + \frac{Q_i^\theta}{R} + q^i_\theta = \left( T_i^1 v_1^i + T_2^i \psi_1^i \right),
\]
(2b)

\[
\frac{\partial Q_i^i}{\partial z} + \frac{\partial Q_{i\theta}^i}{\partial \theta} - \frac{N_i^i}{R} + q^i_\theta = \left( T_i^1 w_1^i \right),
\]
(2c)

\[
\frac{\partial M_i^i}{\partial z} + \frac{\partial M_{i\theta}^i}{\partial \theta} - Q_i^2 + m^i_z = \left( T_2^i u_0^i + T_3^i \psi_3^i \right),
\]
(2d)

\[
\frac{\partial M_i^\theta}{\partial \theta} + \frac{\partial M_{i\theta}^\theta}{\partial \theta} - Q_i^3 + m^i_\theta = \left( T_3^i v_0^i \right).
\]
(2e)

The last two equations (2d) and (2e) are the moment equations at \( \xi = 0 \) along \( \theta \) and \( z \) respectively. Here, \( N_i^i, N_{i\theta}^i, N_{i\theta}^{i\theta} \) and \( N_{i\theta z} \) are in-plane forces per unit length, \( Q_i^\theta \) and \( Q_i^i \) are transverse shear forces per unit length, and \( M_i^i, M_i^\theta, M_{i\theta}^i, M_{i\theta}^\theta \) and \( M_{i\theta z} \) are moments per unit length at \( \xi = 0 \). External forces per unit area acting on the shell \( i \) in the \( \xi, \theta \), and \( r \) directions are noted as \( q^i_\xi, q^i_\theta \), and \( q^i_r \), respectively, while \( m^i_z \) and \( m^i_\theta \) are the external moments per unit area acting on the shell \( i \) at \( \xi = 0 \) along \( \theta \) and \( z \) respectively. Inertia terms \( I_i^k \) are also defined as

\[
I_i^k = \left( I_i^k + \frac{I_i^{k-1}}{R} \right), \quad k = 1, 2, 3,
\]
(3)

with

\[
I_i^k = \rho_i \int_{h_i^+}^{h_i^-} \xi^{k-1} \, d\xi, \quad k = 1, 2, 3, 4.
\]
(4)

Moreover, \( \rho_i \) is the density of the shell \( i \), \( R \) is the radius of the median surface of the cylinder, and \( h_i^- \) and \( h_i^+ \) are the positions of the inner and outer interfaces of the shell \( i \) relative to the reference surface as shown in Fig. 2.

Finally, using the standard force–strain and strain–displacement relations [23], the five equilibrium equations (2a)–(2e) are rewritten in the following form:

\[
L^i u^i + M^i \dot{u}^i = q^i,
\]
(5)

where \( u^i \) is the displacement–rotation vector:

\[
u^i = [u_0^i, v_0^i, w_0^i, \psi_2^i, \psi_3^i]^T.
\]
(6)

\( q^i \) is the force–moment vector:

\[
q^i = [q_\xi^i, q_\theta^i, q_r^i, m_z^i, m_{\theta}^i]^T,
\]
(7)

\( M^i \) is the mass matrix:

\[
M^i = \begin{bmatrix}
T_1^0 & 0 & 0 & T_2^i \\
0 & T_1^1 & 0 & 0 \\
0 & 0 & T_1^3 & 0 \\
T_2^i & 0 & 0 & T_3^i \\
0 & T_2^i & 0 & 0 \\
\end{bmatrix},
\]
(8)

and \( L^i \) is the stiffness operator given in Appendix B.

### 2.2. Transfer matrix of the isotropic core

In order to allow a relative motion between the two skins, the isotropic polymer core is modeled with the standard three-dimensional linear elasticity theory:

\[
\nabla \cdot \sigma = \rho \frac{\partial^2 U}{\partial t^2},
\]
(9)

where \( \sigma \) is the stress tensor, \( \rho \) the density and \( U \) the displacement vector given by \( U = [U_r, U_\theta, U_z]^T \) with \( U_r \), \( U_\theta \) and \( U_z \) being the radial, circumferential and axial components, respectively. By introducing the Lamé parameters \( \lambda \) and \( \mu \), and with a
The classical Helmholtz decomposition of the displacement vector is then used:

\[ U = \nabla \varphi + \nabla \times \psi, \]

(11)

where \( \varphi \) and \( \psi = (\psi_r, \psi_\theta, \psi_z)^T \) are the scalar and vector potential respectively. Substituting this Helmholtz decomposition (11) into Eq. (10) leads to two propagation equations:

\[ \Delta \varphi + \frac{\omega^2}{c_l^2} \varphi = 0, \]

(12a)

\[ \Delta \psi + \frac{\omega^2}{c_t^2} \psi = 0, \]

(12b)

where \( \Delta \) corresponds to the Laplacian operator, and \( c_l \) and \( c_t \) are the longitudinal and transverse wave celerities, respectively, defined by

\[ c_l = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad \text{and} \quad c_t = \sqrt{\frac{\mu}{\rho}}. \]

(13)

Solutions of Eqs. (12a) and (12b) are expanded in cylindrical harmonics:

\[ \varphi(r, \theta, z, t) = \sum_{n=0}^{\infty} \Phi^n(r) \cos(n\theta)e^{ik_nz - j\omega t}, \]

(14a)

\[ \psi_r(r, \theta, z, t) = \sum_{n=0}^{\infty} \Psi^n_r(r) \sin(n\theta)e^{ik_nz - j\omega t}, \]

(14b)

\[ \psi_\theta(r, \theta, z, t) = \sum_{n=0}^{\infty} \Psi^n_\theta(r) \cos(n\theta)e^{ik_nz - j\omega t}, \]

(14c)

\[ \psi_z(r, \theta, z, t) = \sum_{n=0}^{\infty} \Psi^n_z(r) \sin(n\theta)e^{ik_nz - j\omega t}, \]

(14d)

where \( n \) designates the circumferential mode index of the cylinder and \( k_n \) the wavenumber in the axial direction. The expressions of the radial components \( \Phi^n(r) \), \( \Psi^n_r(r) \), \( \Psi^n_\theta(r) \) and \( \Psi^n_z(r) \) are given in Appendix A. The displacement field is then obtained by substituting Eqs. (14a)–(14d) into Eq. (11). This displacement field can be expanded in cylindrical harmonics as follows:

\[ U_r(r, \theta, z, t) = \sum_{n=0}^{\infty} U^n_r(r) \cos(n\theta)e^{ik_nz - j\omega t}, \]

(15a)

\[ U_\theta(r, \theta, z, t) = \sum_{n=0}^{\infty} U^n_\theta(r) \sin(n\theta)e^{ik_nz - j\omega t}, \]

(15b)

\[ U_z(r, \theta, z, t) = \sum_{n=0}^{\infty} j\sigma^n_z(r) \cos(n\theta)e^{ik_nz - j\omega t}, \]

(15c)

where \( U^n_r(r) \), \( U^n_\theta(r) \) and \( U^n_z(r) \) are given in Appendix A. Using the stress–strain and strain–displacement relations, the stress field can be obtained and expanded in cylindrical harmonics in the same manner:

\[ \sigma_{rr}(r, \theta, z, t) = \sum_{n=0}^{\infty} \sigma^n_{rr}(r) \cos(n\theta)e^{ik_nz - j\omega t}, \]

(16a)

\[ \sigma_{r\theta}(r, \theta, z, t) = \sum_{n=0}^{\infty} \sigma^n_{r\theta}(r) \sin(n\theta)e^{ik_nz - j\omega t}, \]

(16b)

\[ \sigma_{rz}(r, \theta, z, t) = \sum_{n=0}^{\infty} j\sigma^n_{rz}(r) \cos(n\theta)e^{ik_nz - j\omega t}. \]

(16c)

Expressions of the radial components \( \sigma^n_{rr}(r) \), \( \sigma^n_{r\theta}(r) \) and \( \sigma^n_{rz}(r) \) are given in Appendix A.

The displacement and stress components at the lower and upper interfaces \( r_1^+ \) and \( r_3^- \) of the core are now related with a Modal Transfer Matrix \( \mathbf{T} \) such as

\[ \mathbf{S}(r_1^+) = \mathbf{T}\mathbf{S}(r_3^-), \]

(17)
with

$$ S(r) = [U_n^r(r), U_r^r(r), \sigma_n^r(r), \sigma_r^r(r), \sigma_{n\theta}^r(r), \sigma_{r\theta}^r(r)]^T. $$

This Modal Transfer Matrix can be partitioned as $T = \begin{bmatrix} A & B \\ D & C \end{bmatrix}$. Calculation of the matrix $T$ must be done with a special care since it necessitates the inversion of a matrix $\mathbf{t}(r)$ detailed in Appendix A. This matrix $\mathbf{t}(r)$, relating the displacement–stress vector $S(r)$ to the modal components, can be ill-conditioned for higher order harmonics and at high frequency.

The Fourier components of the reaction forces per unit area $\mathbf{F}_n^i$ and $\mathbf{F}_n^s$, applied by the core respectively on the inner and the outer skin, can then be derived from the Modal Transfer Matrix $T$:

$$ \begin{bmatrix} \mathbf{F}_n^i \\ \mathbf{F}_n^s \end{bmatrix} = \begin{bmatrix} \sigma_n^i \\ -\sigma_n^s \end{bmatrix} = \begin{bmatrix} DB^{-1} - CB^{-1}A \\ B^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{U}_n^i \\ \mathbf{U}_n^s \end{bmatrix}, $$

where $\mathbf{U}_n^i = [U_n^r(r_1^+), U_n^r(r_1^-), U_n^r(r_3^-)]^T$ and $\mathbf{U}_n^s = [U_n^r(r_3^+), U_n^r(r_3^-), U_n^r(r_3^-)]^T$ are the displacements at $r_1^+$ and $r_3^-$ respectively, while $\sigma_n^i = [\sigma_n^{r}(r_1^+), \sigma_n^{n}(r_1^+), \sigma_n^s(r_1^+)]^T$ and $\sigma_n^s = [\sigma_n^{r}(r_3^-), \sigma_n^{n}(r_3^-), \sigma_n^s(r_3^-)]^T$ are the stress components at $r_1^+$ and $r_3^-$ respectively. The matrix $k$ in Eq. (19) corresponds to the core dynamic stiffness matrix.

2.3. Interlaminar continuity conditions

(i) The continuity of the displacements must be satisfied at the core–skin interfaces. At $r = r_1^+$ (interface between layers 1 and 2) this condition writes

$$ U_i(r_1^+, \theta, z) = u^i(z, \theta, h_1^+) = u_0^i(z, \theta) + h_1^+ \psi_2^i(z, \theta), $$(20a)

$$ U_\theta(r_1^+, \theta, z) = v^i(z, \theta, h_1^+) = v_0^i(z, \theta) + h_1^+ \psi_3^i(z, \theta), $$(20b)

$$ U_r(r_1^+, \theta, z) = w^i(z, \theta, h_1^+) = w_0^i(z, \theta), $$(20c)

while at $r = r_3^-$ (interface between layers 2 and 3) it writes

$$ U_i(r_3^-, \theta, z) = u^i(z, \theta, h_3^-) = u_0^i(z, \theta) + h_3^- \psi_2^i(z, \theta), $$(21a)

$$ U_\theta(r_3^-, \theta, z) = v^i(z, \theta, h_3^-) = v_0^i(z, \theta) + h_3^- \psi_3^i(z, \theta), $$(21b)

$$ U_r(r_3^-, \theta, z) = w^i(z, \theta, h_3^-) = w_0^i(z, \theta). $$(21c)

(ii) Instead of using the stress continuity explicitly, the forces $\mathbf{q}_i^i$ appearing in the right-hand side of the skins equation (5) will be split as the sum of the reaction forces $\mathbf{q}_i^{\text{core}}$ applied by the core on the skin $i$ and the external forces $\mathbf{q}_i^\text{ext}:

$$ L^i \mathbf{u} + M^i \ddot{\mathbf{u}} = \mathbf{q}_i^{\text{core}} + \mathbf{q}_i^\text{ext}. $$

The generalized core reaction forces given here are obtained by using the stress components of the core in Eq. (16) such as

$$ \mathbf{q}_i^{\text{core}} = \begin{bmatrix} \sigma_{rz}(r_1^+, \theta, z, t) \\ \sigma_{r\theta}(r_1^+, \theta, z, t) \\ \sigma_{n\theta}(r_1^+, \theta, z, t) \\ h_1^+ \sigma_{rz}(r_1^+, \theta, z, t) \\ h_1^+ \sigma_{r\theta}(r_1^+, \theta, z, t) \end{bmatrix} \quad \text{and} \quad \mathbf{q}_i^{\text{core}}^3 = \begin{bmatrix} \sigma_{rz}(r_3^-, \theta, z, t) \\ \sigma_{r\theta}(r_3^-, \theta, z, t) \\ \sigma_{n\theta}(r_3^-, \theta, z, t) \\ h_3^- \sigma_{rz}(r_3^-, \theta, z, t) \\ h_3^- \sigma_{r\theta}(r_3^-, \theta, z, t) \end{bmatrix}, $$

while the external forces $\mathbf{q}_i^\text{ext}$ write

$$ \mathbf{q}_i^\text{ext} = [f_{x,\text{ext}}^i, f_{y,\text{ext}}^i, f_{z,\text{ext}}^i, m_{x,\text{ext}}^i, m_{y,\text{ext}}^i]^T, $$

with $f_{x,\text{ext}}^i, f_{y,\text{ext}}^i$ and $f_{z,\text{ext}}^i$ being the external forces per unit area, and $m_{x,\text{ext}}^i$ and $m_{y,\text{ext}}^i$ the external moments per unit area.

2.4. Global dynamic equilibrium

The two equations of motion of the skins are now grouped into a single system:

$$ \begin{bmatrix} L^1 & 0 \\ 0 & L^3 \end{bmatrix} \begin{bmatrix} \mathbf{u}^1 \\ \mathbf{u}^3 \end{bmatrix} + \begin{bmatrix} M^1 & 0 \\ 0 & M^3 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}^1 \\ \ddot{\mathbf{u}}^3 \end{bmatrix} = \begin{bmatrix} \mathbf{q}_i^{\text{core}} \\ \mathbf{q}_i^{\text{core}}^3 \end{bmatrix} + \begin{bmatrix} \mathbf{q}_i^\text{ext} \\ \mathbf{q}_i^\text{ext} \end{bmatrix}, $$

(25)
and, as in Section 2.2, the skins displacements are expanded in cylindrical harmonics:

\[
\begin{bmatrix}
    u_0 \\
v_0 \\
w_{0z} \\
\psi_{0z}
\end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix}
    j u_{0n} \cos(n\theta) \\
v_{0n} \sin(n\theta) \\
w_{0n} \cos(n\theta) \\
\psi_{0n} \sin(n\theta)
\end{bmatrix} e^{jk\nu_z - j\omega t}.
\]

(26)

as well as the external forces:

\[
\begin{bmatrix}
f_{z,\text{ext}} \\
f_{r,\text{ext}} \\
m_{r,\text{ext}} \\
m_{\nu,\text{ext}}
\end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix}
   j f_{z,\text{ext}} \cos(n\theta) \\
f_{r,\text{ext}} \sin(n\theta) \\
f_{r,\text{ext}} \cos(n\theta) \\
m_{\nu,\text{ext}} \sin(n\theta)
\end{bmatrix} e^{jk\nu_z - j\omega t}.
\]

(27)

Since the trigonometric functions \(\cos(n\theta)\) and \(\sin(n\theta)\) are orthogonal along the cylinder’s circumference, equation of motion (25) can be rewritten for each circumferential mode \(n\) as follows:

\[
\begin{bmatrix}
    K^{i} & 0 \\
    0 & K^{j}
\end{bmatrix}
\begin{bmatrix}
    u_{n}^{i} \\
u_{n}^{j}
\end{bmatrix} - \omega^2 \begin{bmatrix}
    M^{i} & 0 \\
    0 & M^{j}
\end{bmatrix}
\begin{bmatrix}
    u_{n}^{i} \\
u_{n}^{j}
\end{bmatrix} =
\begin{bmatrix}
    q_{n,\text{core}}^{i} \\
q_{n,\text{core}}^{j}
\end{bmatrix} +
\begin{bmatrix}
    q_{n,\text{ext}}^{i} \\
q_{n,\text{ext}}^{j}
\end{bmatrix},
\]

(28)

where \(u_{n}^{i}\) is the displacement–rotation amplitude vector:

\[
u_{n}^{i} = [u_{0n}, v_{0n}, w_{0n}, \psi_{0n}]^{T},
\]

\(q_{n,\text{core}}^{i}\) is the generalized reaction amplitude vector:

\[
q_{n,\text{core}}^{i} = \begin{bmatrix}
    \sigma_{r0}^{n}(r_{1+}) \\
    \sigma_{\theta 0}^{n}(r_{1+}) \\
    \sigma_{z0}^{n}(r_{1+}) \\
    h_{1+} \sigma_{r0}^{n}(r_{1+})
\end{bmatrix}
\]

and \(q_{n,\text{core}}^{j}\) is:

\[
q_{n,\text{core}}^{j} = \begin{bmatrix}
    \sigma_{r0}^{n}(r_{3-}) \\
    \sigma_{\theta 0}^{n}(r_{3-}) \\
    \sigma_{z0}^{n}(r_{3-}) \\
    h_{3-} \sigma_{r0}^{n}(r_{3-})
\end{bmatrix}.
\]

(30)

\(q_{n,\text{ext}}^{i}\) is the external force amplitude vector:

\[
q_{n,\text{ext}}^{i} = [f_{z,\text{ext}}^{i}, f_{r,\text{ext}}^{i}, m_{r,\text{ext}}^{i}, m_{\nu,\text{ext}}^{i}]^{T}.
\]

(31)

and \(K^{i}\) is the stiffness matrix given in Appendix B. Note that the stiffness matrix \(K^{i}\) of the skins is symmetric thanks to the cylindrical harmonics expansion used for the axial displacement terms \(u_{0}\) and \(\psi_{0}\) where the Fourier components have been multiplied, as in Section 2.2, by the complex number \(j\).

The generalized reaction amplitude vectors \(q_{n,\text{core}}^{i}\) and \(q_{n,\text{core}}^{j}\) in the right-hand side of Eq. (28) are linear functions of the skins displacements. First, the stress components of the core in Eq. (30) are expressed with the core displacements by using the core dynamic stiffness matrix \(K\) in Eq. (19). Then, the core displacements at the interfaces are replaced by the skins displacements by using the continuity conditions (20) and (21). The resulting generalized reaction amplitude vectors write hence:

\[
\begin{bmatrix}
    q_{n,\text{core}}^{i} \\
q_{n,\text{core}}^{j}
\end{bmatrix} =
\begin{bmatrix}
    K_{11}^{i} & K_{13}^{i} \\
    K_{31}^{j} & K_{33}^{j}
\end{bmatrix}
\begin{bmatrix}
    u_{n}^{i} \\
u_{n}^{j}
\end{bmatrix},
\]

(32)

with \(K_{11}^{i}\) given in Appendix C.

Finally, substituting Eq. (32) in Eq. (28) yields

\[
\begin{bmatrix}
    K^{i} - K_{11}^{i} & -K_{13}^{i} \\
    -K_{31}^{j} & K^{j} - K_{33}^{j}
\end{bmatrix}
\begin{bmatrix}
    u_{n}^{i} \\
u_{n}^{j}
\end{bmatrix} - \omega^2 \begin{bmatrix}
    M^{i} & 0 \\
    0 & M^{j}
\end{bmatrix}
\begin{bmatrix}
    u_{n}^{i} \\
u_{n}^{j}
\end{bmatrix} =
\begin{bmatrix}
    q_{n,\text{ext}}^{i} \\
q_{n,\text{ext}}^{j}
\end{bmatrix}.
\]

(33)

Eq. (33) describes the motion of the entire structure excited by external forces. This equation shows clearly the coupling between the inner and the outer skin with the impedance matrix of the core \(K^{j}(\omega)\).
3. Vibroacoustic problem

3.1. Acoustic pressures in the fluid domains

The cylinder is excited by an external oblique plane wave. This incident pressure is expanded in cylindrical harmonics:
\[ p^I(r, \theta, z, t) = p_0 \sum_{n=0}^{\infty} \varepsilon_n J_n(k_n \gamma) \cos(n \theta) \exp(ik_n z - j \omega t), \]  
where \( p_0 \) is the amplitude, \( J_n \) is the Bessel function of the first kind of order \( n \), and \( \varepsilon_n \) is the Neumann factor (\( \varepsilon_n = 1 \) if \( n = 0 \), \( \varepsilon_n = 2 \) if \( n \neq 0 \)). The radial and axial wavenumbers in the exterior fluid, \( k_{1r} \) and \( k_{1z} \), respectively, are given by
\[ k_{1r} = k_1 \cos \gamma \quad \text{and} \quad k_{1z} = k_1 \sin \gamma, \]
with \( \gamma \) being the incidence angle and \( k_1 = \omega/c_1 \) is the acoustic wavenumber, where \( c_1 \) is the speed of sound in the exterior fluid. In the same way, the external pressure diffracted by the cylinder is given by
\[ p^D(r, \theta, z, t) = \sum_{n=0}^{\infty} A_n H_n^1(k_n r) \cos(n \theta) \exp(ik_n z - j \omega t), \]
where \( A_n \) is the unknown amplitude of the diffracted wave, and \( H_n^1 \) is the Hankel function of the first kind and order \( n \).

For a resonant interior cavity, the transmitted internal pressure inside the cylinder \( p^T_{\text{res}} \) is given by
\[ p^T_{\text{res}}(r, \theta, z, t) = \sum_{n=0}^{\infty} B_n J_n(k_n r) \cos(n \theta) \exp(ik_n z - j \omega t), \]
where \( B_n \) is the unknown amplitude of the transmitted wave in the case of a non-resonant interior cavity, and \( k_{2r} \) and \( k_{2z} \) are the radial and axial wavenumbers in the interior fluid, respectively.

In order to compare with the results published in Ref. [18] which consider the interior domain as non-resonant, we restrict our study to this particular case. Thus, with a non-resonant cavity, the transmitted internal pressure writes
\[ p^T(r, \theta, z, t) = \sum_{n=0}^{\infty} B_n H_n^1(k_n r) \cos(n \theta) \exp(ik_n z - j \omega t), \]
where \( B_n \) is the unknown amplitude of the transmitted wave in the case of a non-resonant interior cavity, and \( H_n^1 \) is the Hankel function of the second kind and order \( n \). Moreover, all the waves have the same dependence in the axial direction. The axial wavenumber in the interior fluid \( k_{2z} \) is thus expressed as
\[ k_{2z} = k_{1z} = k_2. \]
Finally, the radial wavenumber in the interior fluid \( k_{2r} \) is simply given by
\[ k_{2r}^2 = k_{1r}^2 + k_{2z}^2, \]
with \( k_2 = \omega/c_2 \), and where \( c_2 \) is the speed of sound in the interior fluid.

3.2. Boundary conditions

On the cylinder surfaces, the radial velocity continuity between the fluid and the structure must be verified:
\[ \frac{\partial p_1}{\partial r} \bigg|_{r = r_s} = -\rho_1 \frac{\partial^2 w_1^3}{\partial z^2}, \]  
\[ \frac{\partial p_2}{\partial r} \bigg|_{r = r_i} = -\rho_2 \frac{\partial^2 w_2^3}{\partial z^2}, \]
where \( p_1 = p^I + p^D \) and \( p_2 = p^T \) are the total pressures in the external and internal fluids, respectively, and where \( \rho_1 \) and \( \rho_2 \) are the fluid densities in the exterior and interior domains, respectively.

The diffracted and transmitted modal amplitudes \( A_n \) and \( B_n \) are hence related to the radial skins displacements amplitudes \( w_{0n}^3 \) and \( w_{1n}^3 \):
\[ A_n = \frac{\rho_1 \omega^2 w_{0n}^3 - \rho_0 \varepsilon_n J_n(k_{1r} r_s)}{H_n^1(k_{1r} r_s)} k_{1r}, \]  
\[ B_n = \frac{\rho_2 \omega^2 w_{1n}^3}{H_n^1(k_{2r} r_i)} k_{2r}, \]
where the prime denotes the derivative with respect to the argument. The external and internal pressures \( p_1 \) and \( p_2 \) are now decomposed into blocked-wall pressures and radiated pressures, and expressed with the skins displacements \( w_{0n}^3 \) and \( w_{1n}^3 \). For the exterior domain it writes
where

\[ p_1 = p^b + p_1^{rad}, \]  

(43)

is the blocked-wall pressure, which corresponds to \( p^l + p^k \) for \( w^3 = 0 \), and where

\[ p_1^{rad} = \sum_{n=0}^{\infty} \rho_1 n^2 H_n^1(k_1 r) H_n^1(k_1 r +) \cos(n \theta) e^{i k_1 r - j \omega t} \]  

(44)

is the pressure radiated by the cylinder in the exterior medium. Similarly for the interior domain, we obtain

\[ p_2 = p_2^{rad} = \sum_{n=0}^{\infty} \rho_2 n^2 H_n^2(k_2 r) W_0^n \cos(n \theta) e^{i k_2 r - j \omega t}, \]  

(45)

(46)

\[ Z_1^{rad} \] and \( Z_2^{rad} \) are the radiation impedance of the external and internal faces of the cylinder, respectively, and are given by

\[ p_1^{rad} = \frac{\rho_1 n^2 H_n^1(k_1 r +) H_n^1(k_1 r +)}{H_n^1(k_1 r +) k_1 r}, \]  

(49)

\[ p_1^{rad} = \frac{\rho_1 n^2 H_n^1(k_1 r +) H_n^1(k_1 r +)}{H_n^1(k_1 r +) k_1 r}, \]  

(50)

\[ p_2^{rad} = \frac{\rho_2 n^2 H_n^2(k_2 r +) H_n^2(k_2 r +)}{H_n^2(k_2 r +) k_2 r}, \]  

(51)

Note that the radiated pressures \( p_1^{rad} \) and \( p_2^{rad} \) can be written in the following form:

\[ p_1^{rad} = Z_{1n} W_0^n \quad \text{and} \quad p_2^{rad} = Z_{2n} W_0^n, \]  

(52)

where \( Z_{1n} \) and \( Z_{2n} \) are the radiation impedance of the external and internal faces of the cylinder, respectively, and are given by

\[ Z_{1n} = \frac{\rho_1 n^2 H_n^1(k_1 r +)}{H_n^1(k_1 r +) k_1 r} \quad \text{and} \quad Z_{2n} = \frac{\rho_2 n^2 H_n^2(k_2 r +)}{H_n^2(k_2 r +) k_2 r}, \]  

(53)

The external forces \( q_{1, \text{ext}}^l \) and \( q_{3, \text{ext}}^l \) applied to the cylinder are then written as

\[ \begin{bmatrix} q_{1, \text{ext}}^l \\ q_{3, \text{ext}}^l \end{bmatrix} = \begin{bmatrix} Z_1^l & 0 \\ 0 & Z_2^l \end{bmatrix} \begin{bmatrix} u_1^l \\ u_3^l \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} p_1^b \\ 0 \end{bmatrix}, \]  

(54)

where

\[ q_{1, \text{ext}}^l = [0, 0, -p_n^b, 0, 0]^T, \]  

(55)

\[ Z_1^l = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Z_{2n} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Z_2^l = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -Z_{1n} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \]  

(56)

3.3. Global vibroacoustic system

The coupling between the structure and the surrounding air is simply made in the right-hand side of Eq. (33) with the following external forces:

\[ q_{1, \text{ext}}^l = [0, 0, p_2^{rad}, 0, 0]^T, \]  

(47)

\[ q_{3, \text{ext}}^l = [0, 0, -(p_n^b + p_1^{rad}), 0, 0]^T, \]  

(48)

where \( p_n^b, p_1^{rad} \) and \( p_2^{rad} \) are the cylindrical harmonics of the pressures \( p^b, p_1^{rad} \) and \( p_2^{rad} \), respectively, given by

\[ p_1^{rad} = \sum_{n=0}^{\infty} \rho_1 n^2 H_n^1(k_1 r) H_n^1(k_1 r +) \cos(n \theta) e^{i k_1 r - j \omega t}, \]  

(44)

(45)

(46)

(47)

(48)

(49)

(50)

(51)

(52)

(53)

(54)

(55)

(56)
Finally, the global vibroacoustic system of equations is obtained by substituting Eq. (54) into Eq. (33):

\[
\begin{bmatrix}
K^1 - K^1_{11} - Z^1
& -K^2_{13}

-K^2_{31} - K^3 - K^3_{33} - Z^3
\end{bmatrix}
\begin{bmatrix}
u^1

\begin{bmatrix}
M^1
0
M^3
\end{bmatrix}

\begin{bmatrix}
u^1
\end{bmatrix}
= \begin{bmatrix} 0 \\
p^b_n \end{bmatrix}.
\]

(57)

The 10 equations governing the vibroacoustic problem are presented in detail in Appendix D.

3.4. Transmission loss calculation

The Transmission Loss (TL) is given by

\[
\text{TL} = 10 \log \frac{W^i}{W^t}
\]

(58)

where \(W^i\) and \(W^t\) are the incident and transmitted powers, respectively. The incident wave being a plane wave, the incident power \(W^i\) is directly obtained, for a unit length, as

\[
W^i = \frac{p^3_{\text{in}} r_3 \cos \gamma}{\rho_1 c_1}.
\]

(59)

The transmitted power \(W^t\) is obtained by integrating the intensity over the cylinder surface having a unit length:

\[
W^t = \frac{1}{2} \text{Re} \left\{ \int_0^{2\pi} \int_0^r p^i(-j \omega w^i)^* r_1 \text{d}z \text{d}\theta \right\} \text{ at } r = r_1.,
\]

(60)

where \(\text{Re}\{\cdot\}\) and \(\ast\) are the real part and the complex conjugate, respectively. Finally, the TL is written as

\[
\text{TL} = -10 \log \sum_{n=0}^{\infty} \frac{\text{Re} \{Z_{2n} w^i_{0n} \cdot (-j \omega w^i_{0n})^* r_1 \rho_1 c_1 \}}{r_3 p^3_{\text{in}} \cos \gamma}. \]

(61)


4. Results

In order to introduce a structural damping \(\eta\) in each layer of the structure, material properties (Young's modulus \(E\) and shear modulus \(G\)) are considered complex:

- for the isotropic core:

\[
E^* = E(1-j\eta_c),
\]

(62)

- for the orthotropic skins:

\[
E^*_a = E_a(1-j\eta_t), \quad a = z, \theta,
\]

\[
G_{ab}^* = G_{ab}(1-j\eta_t), \quad ab = z\theta, zr, \theta r,
\]

(63)

where \(\eta_c\) and \(\eta_t\) are the damping of the core and the damping of the skins, respectively.

In the following, the calculations are made with a polymer core. This kind of material is known to have viscoelastic properties with frequency dependent elastic and damping moduli (see Refs. [24–26]). However, at a given temperature and in a specific frequency range, these moduli can sometimes be assumed as constant, and for the sake of simplicity this case is considered here. Note that this is not a limitation of the model.

4.1. Validation on a sandwich cylinder with a thin core

The results obtained with the proposed mixed 3D-Shell analytical model are compared with those published by Ghinet et al. [18]. In their work, all the layers, even the core, are modeled with the shell theory. To compare with our model, a thin sandwich cylinder with layer thicknesses \(h_1 = h_2 = h_3 = 1\) mm, made of aluminum \(E = 69\) GPa, \(\rho = 2768\) kg/m³, \(\nu = 0.3\), \(\eta = 5\)% and of external radius \(r_3 = 2.164\) m is considered. The structure is surrounded by air with \(\rho_1 = \rho_2 = 1.284\) kg/m³ and \(c_1 = c_2 = 340\) m/s, and is excited by a plane wave with an incidence angle \(\gamma = 45^\circ\). In this case, the two models should give the same results since no relative displacements are possible between the two skins. This is well verified in Fig. 4 where an excellent agreement is visible between the shell model and the mixed 3D-Shell model.
Table 1
Materials properties for the TL calculation.

<table>
<thead>
<tr>
<th>Layer (Material)</th>
<th>Skins (Graphite/Epoxy) [19]</th>
<th>Core (Polymer)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density (kg/m³)</td>
<td>1600</td>
<td>1000</td>
</tr>
<tr>
<td>Young’s modulus $E_z$ (GPa)</td>
<td>137.9</td>
<td>0.001</td>
</tr>
<tr>
<td>Young’s modulus $E_θ$ (GPa)</td>
<td>8.96</td>
<td>0.001</td>
</tr>
<tr>
<td>Shear modulus $G_{zθ}$ (GPa)</td>
<td>7.1</td>
<td>0.00034</td>
</tr>
<tr>
<td>Shear modulus $G_{θr}$ (GPa)</td>
<td>7.1</td>
<td>0.00034</td>
</tr>
<tr>
<td>Shear modulus $G_{zr}$ (GPa)</td>
<td>6.2</td>
<td>0.00034</td>
</tr>
<tr>
<td>Poisson’s ratio $ν_{zθ}$</td>
<td>0.3</td>
<td>0.49</td>
</tr>
<tr>
<td>Damping $η$ (%)</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Thickness h (mm)</td>
<td>2</td>
<td>12</td>
</tr>
</tbody>
</table>

Fig. 4. Transmission Loss comparison between the multilayer shell model in Ref. [18] and the present mixed 3D-Shell analytical model for a 3 mm thick aluminum cylinder ($r_1 = 2.164$ m) excited by a plane wave ($γ=45°$). (---) Mixed 3D-Shell analytical model, (---) multilayer shell model.

Fig. 5. Transmission Loss comparison between the multilayer shell model in Ref. [18] and the present mixed 3D-Shell analytical model for a sandwich cylinder with a thick and soft core detailed in Table 1 and excited by a plane wave ($γ=45°$). (---) Mixed 3D-Shell analytical model, (---) multilayer shell model.

Fig. 6. Deformed shapes of the skins at 75 Hz. On the left: 3D deformed shape, on the right: deformed section at $z=0$. 
4.2. Comparison between the multilayer shell model and the mixed 3D-Shell model for a sandwich cylinder with a thick and soft core

The results are still compared with the multilayer shell model presented in Ref. [18]. A sandwich configuration with a thick and soft core is however considered. The material properties used in the following are presented in Table 1.

The influence of the core modeling on the TL curve is now clearly visible in Fig. 5. The ring frequency $f_r$ (around 100 Hz) separates two domains. (i) Below the ring frequency, the results are identical. In this frequency range, a global motion of the entire structure with no relative radial displacement between the layers is indeed observed. An example of this in-phase motion between the two skins is presented in Fig. 6 at 75 Hz. Consequently, a multilayer shell model is sufficient to predict the frequency response of the sandwich cylinder below the ring frequency. There is therefore no difference between the TL curves obtained by the mixed 3D-Shell and the multilayer shell model. (ii) Above the ring frequency, the behaviors are different. Results obtained with the multilayer shell model show that the TL curve follows the mass law with a slope of 6 dB/octave, up to the coincidence frequency (12 800 Hz) region. On the other hand, with the mixed 3D-Shell model the TL curve exhibits a large dip between 4000 Hz and 10 000 Hz. In this frequency band the skins are vibrating out of phase as shown in Fig. 7. Obviously, this cannot be observed when a multilayer shell model is used since all the layers move together without relative radial displacements.

![Deformed shapes of the skins at 4300 Hz. On the left: 3D deformed shape, on the right: deformed section at $z=0$.](image1)

![Influence of the core damping $\eta_c$ on the Transmission Loss of a sandwich cylinder with a thick and soft core detailed in Table 1 and excited by a plane wave ($\gamma=45^\circ$). Calculations made with the mixed 3D-Shell analytical model.](image2)

![Influence of the skins damping $\eta_s$ on the Transmission Loss of a sandwich cylinder with a thick and soft core detailed in Table 1 and excited by a plane wave ($\gamma=45^\circ$). Calculations made with the mixed 3D-Shell analytical model.](image3)
4.3. Parametric study

In this part, the mixed 3D-Shell analytical model is used to study the influence of the damping on the TL curve on the same configuration presented in Table 1.

Fig. 8 shows the TL curves for two core damping ratios ($\eta_c$): 0.5% and 5%. Once again, the ring frequency $f_r$ separates two domains. (i) Below the ring frequency, the core damping has no influence on the TL since the skins are vibrating in phase leading to no radial deformation of the core layer. (ii) Above the ring frequency, the core damping has an observable effect by increasing the TL level at the curve dips. This is particularly visible between 4000 Hz and 10 000 Hz where the skins move out of phase. In addition, the core damping smooths the TL curve at the structural resonances, for instance at 761 Hz, 2290 Hz or 3800 Hz.

Fig. 9 shows TL curves obtained with a core damping ratio ($\eta_c$) of 5% and using two values of skin damping ratios ($\eta_s$): 0% and 5%. This time, the skins damping increases the TL at the dips even below the ring frequency.

4.4. Strain energy

This section presents a strain energy analysis made on a sandwich cylinder to understand its dynamic behavior.

The strain energy of the core per unit length is given by

$$ U_c = \frac{1}{2} \text{Re} \left( \frac{1}{2} \int_0^1 \int_0^{2\pi} \int_{r_1}^{r_2} \sigma_c^T (\varepsilon_c)^* r \, dr \, d\theta \, dz \right), \tag{64} $$

where

$$ \{\sigma_c\} = \{\sigma_{zz}, \sigma_{\theta\theta}, \sigma_{rr}, \sigma_{r\theta}, \sigma_{zr}\}^T, \tag{65} $$

and

$$ \{\varepsilon_c\} = \{\varepsilon_{zz}, \varepsilon_{\theta\theta}, \varepsilon_{rr}, \gamma_{r\theta}, \gamma_{zr}\}^T. \tag{66} $$

Similarly, the strain energy for each skin writes

$$ U_i = \frac{1}{2} \text{Re} \left( \frac{1}{2} \int_0^1 \int_0^{2\pi} \int_{h_{i-}}^{h_i} \sigma_i^T (\varepsilon_i)^* (R + \xi) d\xi \, d\theta \, dz \right), \tag{67} $$

where

$$ \{\sigma_i\} = \{\sigma_{zz}^i, \sigma_{\theta\theta}^i, \sigma_{rr}^i, \sigma_{r\theta}^i, \sigma_{zr}^i\}^T, \tag{68} $$

and

$$ \{\varepsilon_i\} = \{\varepsilon_{zz}^i, \varepsilon_{\theta\theta}^i, \varepsilon_{rr}^i, \gamma_{r\theta}^i, \gamma_{zr}^i\}^T. \tag{69} $$

Fig. 10 shows the strain energy of each layer (the core, inner and outer skins). Below 400 Hz, the strain energy of the core is lower than the skins energies since, as highlighted in the previous section, in this lower frequency band the skins are vibrating in phase leading to a small deformation of the core layer. On the other hand, above 400 Hz, the strain energy of the core is higher than those of the skins.

In order to explain the observed frequency behavior, the strain energy will be decomposed into compression and shear terms. The compression $U_{c\text{comp}}$ and shear $U_{c\text{shear}}$ terms are respectively computed for the core as follows:

$$ U_{c\text{comp}} = \frac{1}{4} \text{Re} \left( \int_0^1 \int_0^{2\pi} \int_{r_1}^{r_2} (\sigma_{zz}^c \varepsilon_{zz}^c + \sigma_{\theta\theta}^c \varepsilon_{\theta\theta}^c + \sigma_{rr}^c \varepsilon_{rr}^c) r \, dr \, d\theta \, dz \right), \tag{70} $$

Fig. 10. Comparison of the total strain energy in each layer for a sandwich cylinder with a thick and soft core detailed in Table 1 and excited by a plane wave ($\gamma=45^\circ$). Calculations made with the mixed 3D-Shell analytical model. (---) Core, (-----) inner skin, (----) outer skin.
Similarly, the compression $U_{c,comp}$ and shear $U_{c,shear}$ terms are respectively computed for the skins as follows:

$$U_{c,comp} = \frac{1}{4} \text{Re} \left( \int_0^1 \int_0^{2\pi} \int_{r_1}^{r_2} r (\sigma_{r\theta r\theta}^* + \sigma_{rz rz}^* + \sigma_{z r z r}) R \ dr \ d\theta \ dz \right). \quad (71)$$

Similarly, the compression $U_{s,comp}$ and shear $U_{s,shear}$ terms are respectively computed for the skins as follows:

$$U_{s,comp} = \frac{1}{4} \text{Re} \left( \int_0^1 \int_0^{2\pi} \int_{h_1}^{h_2} (\sigma_{z r z r}^* + \sigma_{\theta r \theta r}^* + \sigma_{\theta z \theta z}^*) (R + \xi) \ d\xi \ d\theta \ dz \right). \quad (72)$$

$$U_{s,shear} = \frac{1}{4} \text{Re} \left( \int_0^1 \int_0^{2\pi} \int_{h_1}^{h_2} (\sigma_{\theta r \theta r}^* + \sigma_{\theta z \theta z}^* + \sigma_{r z r z}^*) (R + \xi) \ d\xi \ d\theta \ dz \right). \quad (73)$$

**Fig. 11.** Comparison of the strain energy contributions for the core of a sandwich cylinder with a thick and soft core detailed in Table 1 and excited by a plane wave ($\gamma=45^\circ$). Calculations made with the mixed 3D-Shell analytical model. (—) $U_c$ (total strain energy in the core), (—–) $U_{c,comp}$, (——) $U_{c,shear}$. 

**Fig. 12.** Comparison of the strain energy contributions for the outer skin of a sandwich cylinder with a thick and soft core detailed in Table 1 and excited by a plane wave ($\gamma=45^\circ$). Calculations made with the mixed 3D-Shell analytical model. (—) $U_s^3$ (total strain energy in the outer skin), (—–) $U_{s,comp}^3$, (——) $U_{s,shear}^3$. 

Fig. 11 shows the compression term $U_{c,comp}$, shear term $U_{c,shear}$, and the total energy $U_c$ of the core. Two frequency bands can be clearly distinguished. Below 148 Hz the shear term $U_{c,shear}$ is dominating. In this frequency range the skins are vibrating in phase and the core motion is imposed by the skins also dominated by the shear component (see Fig. 12). Above 148 Hz the compression term $U_{c,comp}$ is dominating, except at the three resonances located at 761 Hz, 1520 Hz and 2290 Hz where the shear term is locally higher than the compression term. Above 4000 Hz the skins are moving out of phase and the core is therefore subjected to radial relative displacements leading to a higher compression component in the strain energy.

Fig. 12 shows the compression term $U_{s,comp}^3$, shear term $U_{s,shear}^3$ and total energy $U_s^3$ of the outer skin. Two frequency bands can also be distinguished: below 73 Hz, close to the ring frequency (100 Hz), where the shear term $U_{s,shear}^3$ is slightly dominating, and above 73 Hz where the compression term $U_{s,comp}^3$ is slightly dominating. Indeed, the compression and shear terms remain in the same order of magnitude. The strain energies for the inner skin are similar to the outer skin energies and are therefore not presented.
5. Conclusion

A mixed 3D-Shell analytical model has been developed to calculate the sound transmission through an infinite sandwich cylinder. It has been successfully applied to a cylinder composed of two thin orthotropic skins and a thick isotropic core. The First-order Shear Deformation Theory (FSDT) has been used to describe the motion of the two thin orthotropic skins, while the three-dimensional (3D) elasticity theory has been used to describe the motion of the isotropic thick polymer core. Effects of membrane, bending, transverse shearing and rotational inertia are taken into account in both the inner and outer skins, but the radial deformation in the thickness is not taken into account contrary to the core. The results have been compared with published results where the shell theory was applied in the three layers. Excellent agreements have been observed in the case of a sandwich cylinder with a thin core.

In the case of a thick and soft core, the present model highlighted an out of phase motion between the skins, leading to a strong sound transmission through the cylinder. Such phenomenon cannot be observed with the classical shell theory. The proposed three-dimensional elasticity model for the core is indeed necessary to calculate accurately the TL curve of the sandwich cylinder over the whole frequency band.

The parametric study conducted on structural damping effects has shown that the core damping reduces the noise transmission in medium and high frequency, and that the structural damping of the skins reduces the noise transmission in the low frequency band below the ring frequency of the cylinder.

In conclusion, the proposed mixed 3D-Shell analytical model combining the FSDT for the skins and the 3D elasticity theory for the core is well adapted to describe the dynamic behavior of sandwich cylinders composed of two thin orthotropic skins coupled with an isotropic thick polymer core. Results are valid over all the frequencies, and can be used efficiently to predict the noise transmitted inside sandwich cylinders with a thick polymer core. Finally, the proposed model will be extended in the near future to sandwich cylinders with thick poroelastic cores [27,28].

Acknowledgments

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Appendix A. Additional data for the core vibration

(i) The radial components $\Phi^r(r)$, $\Psi^r_{n}(r)$, $\Psi^\theta_{n}(r)$ and $\Psi^\phi_{n}(r)$ in Eqs. (14a)–(14d) are given by

\[
\Phi^r(r) = A_1J_n(q_1 r) + A_2 Y_n(q_1 r), \quad (A.1a)
\]

\[
\Psi^r_{n}(r) = \frac{j k_n}{q_3^2} (B_1 J_{n-1}(q_3 r) + B_2 Y_{n-1}(q_3 r)) + \frac{2n}{q_3^2} (C_1 J_{n}(q_3 r) + C_2 Y_{n}(q_3 r)), \quad (A.1b)
\]

\[
\Psi^\theta_{n}(r) = \frac{j k_n}{q_3^2} (B_1 J_{n-1}(q_3 r) + B_2 Y_{n-1}(q_3 r)) + 2C_1 J_{n}(q_3 r) + 2C_2 Y_{n}(q_3 r), \quad (A.1c)
\]

\[
\Psi^\phi_{n}(r) = B_1 J_{n}(q_3 r) + B_2 Y_{n}(q_3 r), \quad (A.1d)
\]

where $J_n$ and $Y_n$ are respectively the Bessel functions of the first and second kinds, of order $n$, and where

\[
q_1^2 = \frac{\omega^2}{c_1^2} - k_r^2 \quad \text{and} \quad q_3^2 = \frac{\omega^2}{c_3^2} - k_r^2. \quad (A.2)
\]

The six parameters $A_1$, $A_2$, $B_1$, $B_2$, $C_1$ and $C_2$ are unknowns and we have the additional condition $B_1 = B_2 = 0$ for $n=0$. Moreover, the prime denotes the derivative with respect to the argument.

(ii) The radial components of the displacement field given in Eqs. (15a)–(15c) are

\[
U^r_{n}(r) = A_1 q_3^2 J_n(q_1 r) + A_2 q_3^2 Y_n(q_1 r)
\]

\[
+ B_1 \left( \frac{n}{r} J_n(q_3 r) + \frac{k_n^2}{q_3^2} J_{n-1}(q_3 r) \right) + B_2 \left( \frac{n}{r} Y_n(q_3 r) + \frac{k_n^2}{q_3^2} Y_{n-1}(q_3 r) \right) - 2j k_n C_1 J_n(q_3 r) - 2j k_n C_2 Y_n(q_3 r). \quad (A.3a)
\]

\[
U^\theta_{n}(r) = -A_1 \frac{n}{r} J_n(q_1 r) - A_2 \frac{n}{r} Y_n(q_1 r) - B_1 \left( q_3^2 J_n(q_3 r) + \frac{k_n^2}{q_3^2} J_{n-1}(q_3 r) \right) - B_2 \left( q_3^2 Y_n(q_3 r) + \frac{k_n^2}{q_3^2} Y_{n-1}(q_3 r) \right) + 2j k_n C_1 \frac{n}{r} J_n(q_3 r) + 2j k_n C_2 \frac{n}{r} Y_n(q_3 r), \quad (A.3b)
\]

\[
U^\phi_{n}(r) = k_n A_1 J_n(q_1 r) + k_n A_2 Y_n(q_1 r) - k_n B_1 J_n(q_3 r) - k_n B_2 Y_n(q_3 r) + 2j C_1 q_3 J_n(q_3 r) + 2j C_2 q_3 Y_n(q_3 r), \quad (A.3c)
\]

with the additional condition $B_1 = B_2 = 0$ for $n=0$. 


(iii) We can then deduce the radial components of the stress field obtained in Eqs. (16a)–(16c):

\[
\sigma_{rr}^n(r) = A_1 \left( -\frac{\alpha^2}{C_f^2} f_{11}(q_1 r) + 2\mu q_1^2 f_{11}(q_1 r) \right) - 4\mu j_k q_3 C_1 f_{11}(q_3 r) + A_2 \left( -\frac{\alpha^2}{C_f^2} f_{11}(q_1 r) + 2\mu q_1^2 f_{11}(q_1 r) \right) - 4\mu j_k q_3 C_1 f_{11}(q_3 r)
+ 2\mu B_1 \left( \frac{\sigma^2 f_{10}(q_1 r)}{C_0} - \frac{\sigma^2 f_{10}(q_1 r)}{C_1} + j_k^2 f_{11}(q_1 r) \right) + 2\mu B_2 \left( A_1^{(b)} f_{11}(q_1 r) - \frac{\sigma^2 f_{10}(q_1 r)}{C_0} + j_k^2 f_{11}(q_1 r) \right),
\]

(A.4a)

\[
\sigma_{rr}^n(r) = -A_1 \frac{2\mu n}{r} q_1 \left( f_{11}(q_1 r) - \frac{f_{10}(q_1 r)}{q_1 r} \right) - A_2 \frac{2\mu n}{r} q_1 \left( f_{11}(q_1 r) - \frac{f_{10}(q_1 r)}{q_1 r} \right) - B_1 \mu \left( q_3^2 f_{11}(q_3 r) + 2f_{11}(q_3 r) + j_k^2 f_{11}(q_1 r) \right)
+ B_2 \mu \left( q_3^2 f_{10}(q_3 r) + 2f_{10}(q_3 r) + j_k^2 f_{11}(q_1 r) \right) + 4\mu j_k C_1 \frac{n}{n_1} \left( f_{11}(q_3 r) - \frac{f_{10}(q_3 r)}{q_1 r} \right) + 4\mu j_k C_1 \frac{n}{n_1} \left( f_{11}(q_3 r) - \frac{f_{10}(q_3 r)}{q_1 r} \right),
\]

(A.4b)

\[
\sigma_{rr}^n(r) = 2\mu c_k A_1 q_1 f_{11}(q_1 r) + 2\mu c_k A_2 q_1 f_{11}(q_1 r) + B_1 \frac{k_{11}^2}{q_3} \left( q_3^2 f_{11}(q_3 r) + 2f_{11}(q_3 r) + j_k^2 f_{11}(q_1 r) \right) - 2\mu c_k \left( k_{11}^2 - q_3^2 \right) f_{11}(q_3 r)
+ B_2 \frac{k_{11}^2}{q_3} \left( q_3^2 f_{10}(q_3 r) + 2f_{10}(q_3 r) + j_k^2 f_{11}(q_1 r) \right) - 2\mu c_k \left( k_{11}^2 - q_3^2 \right) f_{11}(q_3 r),
\]

(A.4c)

with the additional condition \( B_1 = B_2 = 0 \) for \( n = 0 \).

(iv) In a matrix form, the radial components of the displacement and stress fields are written as

\[
\mathbf{S}(r) = [t(r)] \mathbf{X},
\]

(A.5)

with \( \mathbf{X} = [A_1, A_2, B_1, B_2, C_1, C_2]^T \) and \( \mathbf{S}(r) = [U_{11}^n(r), U_{10}^n(r), U_{11}^n(r), \sigma_{rr}^n(r), \sigma_{rr}^n(r)]^T \). The radial components at the inner interface of the core \( (r = r_{1+}) \) can then be deduced:

\[
\mathbf{S}(r_{1+}) = [t(r_{1+})] \mathbf{X},
\]

(A.6)

and similarly at the outer interface \( (r = r_{3-}) \):

\[
\mathbf{S}(r_{3-}) = [t(r_{3-})] \mathbf{X}.
\]

(A.7)

By substituting the unknowns vector \( \mathbf{X} \) in the last two equations, the relation between the inner and outer interfaces can easily be deduced:

\[
\mathbf{S}(r_{1+}) = [t(r_{1+})] [t(r_{3-})]^{-1} \mathbf{S}(r_{3-}).
\]

(A.8)

Finally, the transfer matrix \( \mathbf{T} \), expressed in Eq. (17), is found as

\[
\mathbf{T} = [t(r_{1+})] [t(r_{3-})]^{-1}.
\]

(A.9)

Appendix B. Additional data for the skins vibration

The elements of the stiffness operator \( \mathbf{L}^j \) defined in Eq. (5) are written as

\[
\mathbf{L}_{11}^j = -\frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \theta^2} - \frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \phi^2}.
\]

\[
\mathbf{L}_{12}^j = \mathbf{L}_{21}^j = -\frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \theta^2} - \frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \phi^2}.
\]

\[
\mathbf{L}_{13}^j = -\frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \theta^2} - \frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \phi^2}.
\]

\[
\mathbf{L}_{14}^j = \mathbf{L}_{41}^j = -\frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \theta^2} - \frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \phi^2}.
\]

\[
\mathbf{L}_{15}^j = \mathbf{L}_{51}^j = -\frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \theta^2} - \frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \phi^2}.
\]

\[
\mathbf{L}_{16}^j = \mathbf{L}_{61}^j = -\frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \theta^2} - \frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \phi^2}.
\]

\[
\mathbf{L}_{22}^j = -\frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \theta^2} - \frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \phi^2}.
\]

\[
\mathbf{L}_{23}^j = -\frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \theta^2} - \frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \phi^2}.
\]

\[
\mathbf{L}_{24}^j = \mathbf{L}_{42}^j = -\frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \theta^2} - \frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \phi^2}.
\]

\[
\mathbf{L}_{25}^j = \mathbf{L}_{52}^j = -\frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \theta^2} - \frac{\bar{A}_{11}^j}{\bar{A}_{11}^j} \frac{\partial^2}{\partial \phi^2}.
\]
\[
\begin{align*}
L_{13}^i &= -\mathcal{A}_{55}^i \frac{\partial^2}{\partial z^2} \frac{\mathcal{A}_{44}^i \frac{\partial^2}{\partial \theta^2} + \mathcal{A}_{22}^i}{R^2}, \\
L_{14}^i &= -L_{34}^i = -\mathcal{A}_{55}^i \frac{\partial}{\partial z} \frac{\mathcal{B}_{12}^i}{R} \frac{\partial}{\partial z}, \\
L_{15}^i &= -L_{35}^i = -\frac{\mathcal{A}_{44}^i}{R} \frac{\partial}{\partial \theta} + \frac{\mathcal{B}_{22}^i}{R^2} \frac{\partial^2}{\partial \theta^2}, \\
L_{16}^i &= -\mathcal{D}_{11}^i \frac{\partial^2}{\partial z^2} - \frac{\mathcal{D}_{46}^i}{R^2} \frac{\partial^2}{\partial \theta^2} + \mathcal{A}_{55}^i, \\
L_{45}^i &= L_{54}^i = -\frac{\mathcal{D}_{12}^i}{R} \frac{\partial^2}{\partial z \partial \theta} - \frac{\mathcal{D}_{46}^i}{R} \frac{\partial^2}{\partial \theta^2}, \\
L_{55}^i &= -\frac{\mathcal{D}_{22}^i}{R^2} \frac{\partial^2}{\partial z^2} - \frac{\mathcal{D}_{46}^i}{R^2} \frac{\partial^2}{\partial \theta^2} + \mathcal{A}_{44}^i.
\end{align*}
\]

where
\[
\begin{align*}
\mathcal{A}_{kl}^i &= A_{kl}^i + \frac{B_{kl}^i}{R}, \\
\mathcal{B}_{kl}^i &= \frac{A_{kl}^i - B_{kl}^i}{R}, \\
\mathcal{D}_{kl}^i &= D_{kl}^i + \frac{E_{kl}^i}{R}, \\
\mathcal{D}_{kl}^i &= D_{kl}^i - \frac{E_{kl}^i}{R},
\end{align*}
\]

and
\[
\begin{align*}
A_{kl}^i &= Q_{kl}^i (h_{1+}^i - h_{-1}^i), \\
B_{kl}^i &= Q_{kl}^i \left(\frac{h_{1+}^i + h_{-1}^i}{2}\right), \\
D_{kl}^i &= Q_{kl}^i \left(\frac{h_{1+}^i - h_{-1}^i}{3}\right), \\
E_{kl}^i &= Q_{kl}^i \left(\frac{h_{4+}^i - h_{-4}^i}{4}\right), \\
A_{kl}^i &= K_{kl}^i Q_{kl}^i (h_{1+}^i - h_{-1}^i), \\
B_{kl}^i &= K_{kl}^i Q_{kl}^i \left(\frac{h_{1+}^i - h_{-1}^i}{2}\right),
\end{align*}
\]

and
\[
\begin{align*}
Q_{11}^i &= \frac{E_z^i}{1 - \nu_{12}^i \nu_{21}^i}, & Q_{22}^i &= \frac{E_z^i}{1 - \nu_{12}^i \nu_{21}^i}, \\
Q_{12}^i &= \nu_{12}^i E_z^i \frac{1}{1 - \nu_{12}^i \nu_{21}^i}, & Q_{21}^i &= \nu_{12}^i E_z^i \frac{1}{1 - \nu_{12}^i \nu_{21}^i}, \\
Q_{44}^i &= C_{44}, & Q_{55}^i &= C_{55}, \quad Q_{66}^i = C_{22}^i.
\end{align*}
\]

The \( K_{kl} \) term is the shear correction coefficient, generally taken to be 5/6. Moreover, \( Q_{kl}^i \) are the elastic constants of the skin \( i \) and are defined as

The stiffness term \( K^i \) used in Eq. (28) is obtained from the stiffness operator \( L^i \) and is written as

\[
\begin{align*}
K_{11}^i &= \mathcal{A}_{11}^i k_z^2 + \frac{\mathcal{A}_{66}^i n_z^2}{R^2}, \\
K_{12}^i &= K_{21}^i = -\frac{A_{12}^i + A_{66}^i n_z k_z}{R}, \\
K_{13}^i &= K_{31}^i = -\frac{A_{12}^i k_z}{R}, \\
K_{14}^i &= K_{41}^i = \frac{B_{11}^i k_z^2 + B_{66}^i n_z^2}{R^2}, \\
K_{15}^i &= K_{51}^i = -\frac{B_{12}^i + B_{66}^i n_z k_z}{R},
\end{align*}
\]
\[ K'_{22} = \bar{A}_{66} k_2^2 + \frac{\bar{A}_{22} h^2 + \bar{A}_{44}}{R^2}, \]

\[ K'_{23} = K'_{32} = \frac{\bar{A}_{22} + \bar{A}_{44}}{R^2} n, \]

\[ K'_{24} = K'_{42} = -\frac{B_{12} + B_{66}}{R} n k_2, \]

\[ K'_{25} = K'_{52} = B_{66} k_2^2 + \frac{\bar{B}_{22} h^2}{R^2} n^2 - \frac{\bar{A}_{44}}{R}, \]

\[ K'_{33} = \bar{A}_{55} k_2^2 - \frac{\bar{A}_{44} h^2 + \bar{A}_{22}}{R^2}, \]

\[ K'_{34} = K'_{43} = \left( \frac{\bar{A}_{55} - \frac{B_{12}}{R}}{R} \right) k_2, \]

\[ K'_{35} = K'_{53} = \left( \frac{\bar{B}_{22}}{R^2} + \frac{\bar{A}_{44}}{R} \right) n, \]

\[ K'_{44} = \bar{A}_{55} + D_{11} k_2^2 + \frac{\bar{D}_{66}}{R^2} n^2, \]

\[ K'_{45} = K'_{54} = -\frac{\bar{D}_{12} + \bar{D}_{66}}{R} n k_2, \]

\[ K'_{55} = \bar{A}_{44} + \bar{D}_{66} k_2^2 + \frac{\bar{D}_{22}}{R^2} n^2. \]  (B.5)

**Appendix C. Expressions of submatrices \( K^2_{pq} \)**

The submatrices \( K^2_{pq} \) are defined by

\[
K^2_{11} = \begin{bmatrix}
  k_{11} & k_{12} & k_{13} & h_{1+} k_{11} & h_{1+} k_{12} \\
  k_{21} & k_{22} & k_{23} & h_{1+} k_{21} & h_{1+} k_{22} \\
  k_{31} & k_{32} & k_{33} & h_{1+} k_{31} & h_{1+} k_{32} \\
  h_{1+} k_{11} & h_{1+} k_{12} & h_{1+} k_{13} & (h_{1+})^2 k_{11} & (h_{1+})^2 k_{12} \\
  h_{1+} k_{21} & h_{1+} k_{22} & h_{1+} k_{23} & (h_{1+})^2 k_{21} & (h_{1+})^2 k_{22}
\end{bmatrix},
\]

\[
K^2_{33} = \begin{bmatrix}
  k_{41} & k_{42} & k_{43} & h_{3-} k_{41} & h_{3-} k_{42} \\
  k_{51} & k_{52} & k_{53} & h_{3-} k_{51} & h_{3-} k_{52} \\
  k_{61} & k_{62} & k_{63} & h_{3-} k_{61} & h_{3-} k_{62} \\
  h_{3-} k_{41} & h_{3-} k_{42} & h_{3-} k_{43} & (h_{3-})^2 k_{41} & (h_{3-})^2 k_{42} \\
  h_{3-} k_{51} & h_{3-} k_{52} & h_{3-} k_{53} & (h_{3-})^2 k_{51} & (h_{3-})^2 k_{52}
\end{bmatrix},
\]

\[
K^2_{31} = \begin{bmatrix}
  k_{41} & k_{42} & k_{43} & h_{1+} k_{41} & h_{1+} k_{42} \\
  k_{51} & k_{52} & k_{53} & h_{1+} k_{51} & h_{1+} k_{52} \\
  k_{61} & k_{62} & k_{63} & h_{1+} k_{61} & h_{1+} k_{62} \\
  h_{3-} k_{41} & h_{3-} k_{42} & h_{3-} k_{43} & (h_{3-})^2 k_{41} & (h_{3-})^2 k_{42} \\
  h_{3-} k_{51} & h_{3-} k_{52} & h_{3-} k_{53} & (h_{3-})^2 k_{51} & (h_{3-})^2 k_{52}
\end{bmatrix},
\]

\[
K^2_{33} = \begin{bmatrix}
  k_{44} & k_{45} & k_{46} & h_{3-} k_{44} & h_{3-} k_{45} \\
  k_{54} & k_{55} & k_{56} & h_{3-} k_{54} & h_{3-} k_{55} \\
  k_{64} & k_{65} & k_{66} & h_{3-} k_{64} & h_{3-} k_{65} \\
  h_{3-} k_{44} & h_{3-} k_{45} & h_{3-} k_{46} & (h_{3-})^2 k_{44} & (h_{3-})^2 k_{45} \\
  h_{3-} k_{54} & h_{3-} k_{55} & h_{3-} k_{56} & (h_{3-})^2 k_{54} & (h_{3-})^2 k_{55}
\end{bmatrix},
\]

where the \( k_{ij} \) terms are the elements of the core stiffness matrix \( k \).
Appendix D. Global vibroacoustic equations

The 10 global vibroacoustic equations are

\[
\begin{align*}
&\left(\frac{A_{11}^{-1}k_z^2 + \gamma_{66}^{-1}n^2 - T_1^{-1} \omega^2 - k_{11}}{R^2} \right)u_{0n} + \left( -\frac{A_{12} + A_{66}^{-1}n k_z - k_{12}}{R} \right)v_{0n}' + \left( -\frac{A_{12}^{-1}k_z - k_{13}}{R} \right)w_{0n}' + k_{14}u_{0n}' - k_{15}v_{0n}' - k_{16}w_{0n}' - h_3 - k_{14}\psi_{3n}' - h_3 - k_{15}\psi_{3n}' = 0, \\
&\left( -\frac{A_{12} + A_{66}^{-1}n k_z - k_{21}}{R} \right)u_{1n} + \left( \frac{A_{66}^{-1}k_z^2 + A_{22}^{-1}n^2 + A_{44}^{-1} - T_1^{-1} \omega^2 - k_{22}}{R^2} \right)v_{1n} + \left( \frac{A_{12} + A_{44}^{-1} n - k_{23}}{R} \right)w_{1n}' - k_{24}u_{1n}' - k_{25}v_{1n}' - k_{26}w_{1n}' - h_3 - k_{24}\psi_{3n}' - h_3 - k_{25}\psi_{3n}' = 0, \\
&\left( -\frac{A_{12} + A_{66}^{-1}n k_z - k_{31}}{R} \right)u_{2n} + \left( \frac{A_{66}^{-1}k_z^2 + A_{44}^{-1} n - k_{32}}{R^2} \right)v_{2n} + \left( A_{55}^{-1} - \frac{A_{12}^{-1}}{R} \right)\left( k_z - h_1 + k_{31} \right)w_{2n}' + k_{34}u_{2n}' - k_{35}v_{2n}' - k_{36}w_{2n}' - h_3 - k_{34}\psi_{3n}' - h_3 - k_{35}\psi_{3n}' = 0,
\end{align*}
\]

\[
\begin{align*}
&\left( \frac{A_{66}^{-1}k_z^2 + A_{22}^{-1}n^2 - T_1^{-1} \omega^2 - h_1 + k_{11}}{R^2} \right)u_{0n} + \left( -\frac{B_{12} + B_{66}^{-1}n k_z - h_1 + k_{12}}{R} \right)v_{0n}' + \left( -\frac{B_{12}^{-1}k_z - h_1 + k_{13}}{R} \right)w_{0n}' + k_{14}u_{0n}' - k_{15}v_{0n}' - k_{16}w_{0n}' - h_3 - k_{14}\psi_{3n}' - h_3 - k_{15}\psi_{3n}' = 0, \\
&\left( A_{55}^{-1} - \frac{B_{12}^{-1}}{R} \right)\left( k_z - h_1 + k_{31} \right)w_{0n}' + \left( A_{55}^{-1} + D_{44}^{-1}k_z^2 + D_{66}^{-1}n^2 - T_1^{-1} \omega^2 - (h_1 + \omega^2)k_{11} \right)v_{0n}' + k_{35}u_{0n}' - k_{36}v_{0n}' - h_3 - k_{35}\psi_{3n}' - h_3 - k_{36}\psi_{3n}' = 0, \\
&\left( -\frac{B_{12} + B_{66}^{-1}n k_z - h_1 + k_{21}}{R} \right)u_{1n} + \left( \frac{B_{66}^{-1}k_z^2 + B_{22}^{-1}n^2 - A_{44}^{-1} - T_1^{-1} \omega^2 - h_1 + k_{22}}{R^2} \right)v_{1n}' + \left( -\frac{B_{12} + B_{66}^{-1}n k_z - h_1 + k_{23}}{R} \right)w_{1n}' + k_{24}u_{1n}' - k_{25}v_{1n}' - k_{26}w_{1n}' - h_3 - k_{24}\psi_{3n}' - h_3 - k_{25}\psi_{3n}' = 0,
\end{align*}
\]

\[
\begin{align*}
&\left( A_{44}^{-1} + D_{66}^{-1}k_z^2 + D_{22}^{-1}n^2 - T_1^{-1} \omega^2 - (h_1 + \omega^2)k_{22} \right)v_{1n}' + k_{35}u_{1n}' - k_{36}v_{1n}' - h_3 - k_{35}\psi_{3n}' - h_3 - k_{36}\psi_{3n}' = 0, \\
&\left( -k_{41}u_{0n}' - k_{42}v_{0n}' - k_{43}w_{0n}' - h_1 + k_{42}\psi_{3n}' + \left( A_{11}^{2}k_z^2 + A_{66}^{2}n^2 - T_1^{2} \omega^2 - k_{44} \right)u_{3n}' + k_{35}u_{3n}' - k_{36}v_{3n}' + \left( A_{55}^{2} - \frac{A_{12}^{2}}{R} \right)\left( k_z - k_{46} \right)w_{3n}' + k_{35}u_{3n}' - k_{36}v_{3n}' = 0,
\end{align*}
\]

\[
\begin{align*}
&\left( -k_{41}u_{1n}' - k_{42}v_{1n}' - k_{43}w_{1n}' - h_1 + k_{42}\psi_{3n}' + \left( A_{11}^{2}k_z^2 + A_{66}^{2}n^2 - T_1^{2} \omega^2 - k_{44} \right)u_{5n}' + k_{35}u_{5n}' - k_{36}v_{5n}' + \left( A_{55}^{2} - \frac{A_{12}^{2}}{R} \right)\left( k_z - k_{46} \right)w_{5n}' + k_{35}u_{5n}' - k_{36}v_{5n}' = 0,
\end{align*}
\]
These global vibroacoustic equations have to be solved for each $n$. However, the case $n = 0$ has to be treated carefully. The supplementary condition $B_{1} = B_{2} = 0$ for $n = 0$ in the equations of the isotropic core leads to $U_{00}^{n}(r) = 0$ and $C_{0n}^{n}(r) = 0$, which, therefore, yields $V_{00}^{n} = 0$ and $W_{0n}^{n} = 0$. Consequently, in the case $n = 0$ the global vibroacoustic equations are reduced to a system of six equations with six unknowns, which reflects the fact that the circumferential displacement is null in this case.

References


