The synthesis of spatially correlated random pressure fields

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The feasibility is considered of synthesizing a spatially correlated random pressure field having specified statistical properties. Of particular interest is the use of a near-field array of acoustic sources to synthesize a pressure field whose statistical properties are similar to either a diffuse acoustic sound field or to that generated by a turbulent boundary layer (TBL). A formulation based on least-squares filter design is presented. Initially, the more fundamental question is addressed of how many uncorrelated signal components are required to approximate the pressure field. A one-dimensional analysis suggests that two uncorrelated components per acoustic wavelength are required to approximate a diffuse pressure field. Similarly, for a TBL pressure field, about one uncorrelated component per correlation length is required in the spanwise direction and about two uncorrelated components per correlation length are required in the streamwise direction. These estimates are in good agreement with theoretical predictions for an infinite array, based on the Fourier transform of the spatial correlation function. When a full simulation is performed, including the acoustic effect of an appropriately positioned array of monopole sources, it is found that the number of acoustic sources required to reasonably approximate the diffuse or TBL pressure field is only slightly greater than the lower bound on this number, set by the number of uncorrelated components required. © 2005 Acoustical Society of America. [DOI: 10.1121/1.1850231]

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I. INTRODUCTION

In this article a formulation is presented for using an array of acoustic sources to simulate the spatial correlation properties of a random pressure field, as specified by the spectral density matrix between the elements of an array of acoustic sensors. It is used in the subsequent sections to assess the feasibility of using such an array of sources close to a surface to generate a random pressure field with similar spatial statistical properties to those of either a diffuse acoustic sound field, or the pressure field generated by a turbulent boundary layer.

A diffuse acoustic field is normally defined as being due to an infinite number of uncorrelated plane waves, generated by remote acoustic sources, which, on average, provide equal incident energy from all directions.1,2 It is characterized below in terms of its two-dimensional spatial correlation structure, which is the same in free space or when uncorrelated waves in a half-space fall on an infinite rigid surface.3 Laboratory tests of the sound transmission properties of building structures are often conducted using random noise in a sound transmission suite. The sound field in the reverberant source room of a sound transmission suite is assumed to be diffuse for these measurements, but this is only a reasonable approximation for excitation frequencies above the room’s Schröder frequency.4,5 Below this frequency, individual modes of the source room can dominate the pressure field, which is then not diffuse. The use of a near-field array of acoustic sources, next to the structure being tested, to generate a diffuse pressure field in this low-frequency region has been considered by Bravo and Elliott6 and in the present article we present a general theoretical background that can be used to better understand this problem.

The pressure field generated on the surface of a structure by a turbulent boundary layer (TBL) is also random in both time and space. In a fully developed turbulent boundary layer, the pressure fluctuations over a small region are stationary in time and homogeneous in space and can be characterized by their power spectral density at a point and their cross-spectral density between two separate points.7,8 It is important to be able to measure the response of structures to TBL excitation, particularly in aeronautical applications, and this is currently achieved either with flight tests or in low noise wind tunnels. Both of these test methods are expensive and time consuming. These difficulties have led to the measurement of the acoustic properties of aircraft structures in sound transmission suites, although this is known to underestimate their sound reduction properties when exposed to TBL pressure fluctuations, as will be illustrated below.

Previously, Fahy9 has considered the problem of simu-
lating TBL pressure fluctuations using jet noise, a siren tunnel, a loudspeaker or a shaker, but concluded that none of these single sources could reproduce the decay characteristics or spatial correlation properties in a TBL. Although there is a brief discussion in this paper about the use of an array of shakers, the electrical and mechanical difficulties of implementing such a system were considered, in 1966, to be such that it would be a difficult practical proposition. Robert and Sabot have taken another approach to the laboratory simulation of TBL-excited structures, by noting that the modes of a structure are excited almost independently by such an excitation, and that there is generally a relatively small number of significantly excited structural modes. They considered the use of an array of appropriately-driven shakers acting on the panel, driven so that each structural mode was excited to the required extent. This approach assumes that the mode shapes of the structure are known and so, in practice, some of the required extent. This approach assumes that the mode shapes of the structure are known and so, in practice, some preliminary modal analysis would have to be performed before the drive signals to the shakers could be determined. Dodds has also considered combining together four independent random signals to generate the actuator drive signals in the problem of simulating the partially correlated displacements exciting the four wheels of a vehicle when driven over a rough road. These studies are related to the simulation of multiple channels of partially correlated random data, as described, for example, by Shinozuka, Mignolet and Spanos, and Soize and Poirion. This data is typically used for numerical modelling of the dynamic response of structures, particularly nonlinear structures, to excitations due to the wind, earthquakes, and road excitation. Although the basis of the simulation method varies between these authors, they all begin the analysis with a factorization of the spectral density matrix of the target data, as used later in Sec. IV, and implicitly assume that the multiple channels of the simulated random data are generated by combining together an equal number of uncorrelated random signals.

The direct simulation of a spatially correlated random pressure field with a dense array of acoustic sources is considered here, by arranging for the sources to be mutually correlated so that they reproduce the spatial correlation properties of the target pressure field. The price that must be paid for such a general approach, however, is the large number of sources required for an accurate simulation, particular at high frequencies, as we shall see below. It is shown that the overall problem can be broken down into two parts. The first is concerned with the number of statistically uncorrelated signals required to accurately model the target spectral density matrix, which is related to the proper orthogonal decomposition (POD) of the pressure field. The second part is concerned with the invertibility of a matrix of physical responses between the acoustic sources and the sensors, over which the correlation properties of the target pressure field are reproduced.

II. SPATIAL CORRELATION FUNCTIONS

The spatial correlation structures of a diffuse acoustic field are compared with that of a simple model of a TBL pressure field at high subsonic Mach number, and the different effects that these two pressure fields have in exciting the response of a simple panel is discussed. The spatial correlation structure of the acoustic diffuse field, which is homogeneous and isotropic, leads to an expression for the cross-spectral density between the pressures at two points, A and B, as

$$S_{AB}(\omega) = S_{pp}(\omega) \frac{\sin k_\omega r}{k_\omega r},$$

where $S_{pp}(\omega)$ is the power spectral density of the pressure at any point, $r$ is the distance between the two measurement points, A and B, in any direction and $k_\omega$ is the acoustic wave number given by $k_\omega = \omega/c_0$ where $c_0$ is the speed of sound, which is taken later to be $343 \text{ m s}^{-1}$.

A widely used model for the correlation structure of a fully developed TBL, which will be used for illustration here, is that due to Corcos. This predicts that the cross-spectral density between the pressures at two points, A and B, on a rigid surface is given by

$$S_{AB}(\omega) = S_{pp}(\omega) e^{-|r_x|/L_x} e^{-|r_y|/L_y} e^{-j\omega r_z/U_c},$$

where differences between the $x$ and $y$ coordinates of points A and B are denoted $r_x$ and $r_y$. $S_{pp}(\omega)$ is again the power spectral density at any point, $L_x$ is the correlation length in the spanwise direction, $L_y$ is the correlation length in the streamwise direction, $U_c$ is the convection velocity, and an $e^{j\omega t}$ time dependence has been assumed. It should be emphasized that this is an empirical model and, in contrast to the diffuse field, has not been derived from a theoretical understanding of the underlying physical phenomena. It should also be noted that the simulation of the random pressure field would not reproduce all the physics of a thin plate interacting with a turbulent flow, in particular the aeroelastic–mechanical coupling, as described, for example, by Clark and Frampton, is not included.

The random pressure field is thus modeled as being homogeneous, but nonisotropic, and the Corcos model assumes that the variation of the correlation structure in the spanwise and streamwise directions is independent. In the simulations below the correlation lengths were assumed to be inversely proportional to frequency, and to have the form

$$L_x = \frac{\alpha_x U_c}{\omega},$$

$$L_y = \frac{\alpha_y U_c}{\omega},$$

where $\alpha_x$ and $\alpha_y$ are constants, which were taken to be 1.2 and 8 in the simulations described below. The convection velocity was also assumed to be $135 \text{ m s}^{-1}$, which corresponds to a free-stream velocity of about $225 \text{ m s}^{-1}$ and a Mach number of $M \sim 0.66$. Other authors have suggested different formulations for the correlation lengths in Eqs. (3) and (4), particularly at low frequencies (see, for example, Refs. 16 and 17) but still use the exponential decay of the spatial correlation in Eq. (2). In the limiting case of small $r_x$ and $r_y$, it is known that viscous effects round off the cusp in the exponential peak of the correlation function in Eq. (2), but in the frequency range of interest here, this occurs over a much smaller length scale than the correlation length. The
The normalized spatial correlation structure of the Corcos TBL model in the streamwise and spanwise direction for the conditions above is compared with that for the acoustic diffuse field in Fig. 1.

Figure 2 shows the calculated value of the total kinetic energy of a $0.7 \, \text{m} \times 0.5 \, \text{m} \times 0.003 \, \text{m}$ simply supported aluminum panel, calculated from the sum of the squared mode amplitudes, when subject to either the diffuse or TBL models of the random excitation fields.\textsuperscript{19,20} The streamwise direction of the TBL was assumed to be parallel to the longer dimension of the panel. The power spectral density at any one point in the pressure fields, $S_{pp}(\omega)$ in Eqs. (1) and (2), was assumed to be independent of frequency, and equal in the two cases. It can be seen that the TBL pressure field is considerably less efficient at exciting the vibration of the panel, with the average level of the kinetic energy being about 15 dB.
below that of the diffuse field at low frequencies, and about 20 dB below the diffuse field level at 1500 Hz. These differences in level can partly be explained in terms of the relative widths of the spatial correlation functions shown in Fig. 1, particularly the fact that the width of the spatial correlation function for the TBL in the spanwise direction is about one-tenth that of the diffuse field.19,20

III. DENSITY OF THE UNCORRELATED COMPONENTS

Figure 3 illustrates a one-dimensional array of \( N_S \) acoustic sensors, which in this case are microphones, mounted on a rigid surface and subject either to the pressure field generated by the target random pressure field, left-hand picture, or an array of \( N_A \) acoustic actuators, which are represented as loudspeakers in the right-hand picture. The actuators are assumed to be driven by the output of a matrix of filters, \( W \), which are themselves excited by an array of \( N_R \) reference signals, \( x \). Figure 4 shows the equivalent block diagram for this system, in which \( G \) is the matrix of responses from each actuator to each sensor in Fig. 3, and the original set of desired sensor signals \( d \) is assumed to be generated from the set of reference signals in a way that will be explained in Sec. III.

The statistical properties of the target pressure field at the sensor positions can be characterized by the elements of the spectral density matrix,

\[
S_{dd} = E[dd^H],
\]

where \( d \) is the vector of Fourier transforms, at a single frequency, of one record of the outputs from the microphone array, subject to excitation by the desired random pressure field, \( E \) denotes the expectation operator over the ensemble of records,21 and \( ^H \) denotes the Hermitian, complex conjugate, transpose. The diagonal elements of \( S_{dd} \) correspond to the power spectral densities of the outputs from each individual sensor and the off-diagonal elements to the cross-spectral densities between the outputs of pairs of sensors. The frequency dependence of \( S_{dd} \) and the other variables used below, has been suppressed for notational convenience.

The use of a finite number of discrete measurements allows the problem of approximating the continuous pressure field to be treated using linear algebraic methods, but does assume that the measurements are made on a sufficiently dense grid to accurately sample the correlation properties of the random pressure field under consideration. The number of sensors required to accurately represent diffuse and TBL pressure fields has been discussed by Maury et al.22 It should
be noted, however, that since, in the formulation below, the random pressure field to be reproduced is defined entirely by its spectral density matrix at the sensor positions, the field is not restricted to being homogeneous in the most general case.

In this section a reduced-rank model for the spectral density matrix for the target pressure field, $S_{dd}$, will be considered. If $S_{dd}$, which is an $N_S \times N_S$ matrix, can be approximated to acceptable accuracy with a model of rank $N_U$, then this must be the number of uncorrelated components underlying the pressure field. Moreover, if the number of reference signals is made equal to this number of uncorrelated components, then the rank of the spectral density matrix for the signals driving the actuator array, $S_{uu}$, can be no larger than $N_U$, and so $N_U$ also defines the lower bound on the number of actuators required to reproduce the pressure field to the specified accuracy. In practice it may not be possible to realize such a set of actuators, and when practical actuators are used, a larger number may be required. This is further explored in Sec. IV.

Consider the eigenvalue/eigenvector decomposition, at each frequency, of the Hermitian spectral density matrix for the $N_S$ outputs of the microphone array, excited by the pressure field to be reproduced,

$$S_{dd} = Q \Lambda Q^H,$$

which can also be written as

$$S_{dd} = \sum_{i=1}^{N_S} \lambda_i q_i q_i^H,$$

where $q_i$ are the columns of the eigenvector matrix $Q$, and $\lambda_i$ are the real, positive diagonal elements of the eigenvector matrix $\Lambda$, ordered such that $\lambda_i > \lambda_{i+1}$. This expansion of the spectral density matrix for the discrete set of pressure measurements is analogous to the proper orthogonal decomposition described, for example, by Berkooz et al.\textsuperscript{23}

The best least-squares approximation to $S_{dd}$ with a rank of $N_U$, $\hat{S}_{dd}$, is now obtained by taking the $N_U$ terms with the largest eigenvalues in Eq. (7) and so has the spectral density matrix

$$\hat{S}_{dd} = \sum_{i=1}^{N_U} \lambda_i q_i q_i^H.$$  \hspace{1cm} (8)

Defining a vector of error signals to be

$$e = d - \hat{d},$$

where $\hat{d}$ is the vector of approximated pressure signals, the sum of the mean square errors is equal to

$$J_e = \text{trace}(S_{ee}).$$

The ratio of this sum of mean square errors to the sum of the mean square values of $d$, $J_d = \text{trace}(S_{dd})$, is then equal to

$$\frac{J_e(N_u)}{J_d} = \frac{\sum_{i=N_U+1}^{N_S} \lambda_i}{N_S \sum_{i=1}^{N_S} \lambda_i}.$$  \hspace{1cm} (11)

This expression allows the accuracy of the reduced rank approximations to be quantified. Figure 5 shows the magnitude of the eigenvalues of the $S_{dd}$ matrix calculated from a simulation of a diffuse field measured with 100 microphones in a linear array that extends over ten wavelengths.

![FIG. 5. The eigenvalues of the spectral density matrix calculated for a linear array of 100 microphones exposed to a diffuse acoustic soundfield (dots) together with the theoretical form of this function for an infinite array (solid).](image-url)
trix becomes asymptotically equivalent to a circulant matrix. Since the eigenvectors of a circulant matrix are Fourier components, the eigendecomposition of $S_{dd}$ for a homogeneous pressure field will tend to a spatial Fourier transform. The Fourier transform of the spatial correlation function for an acoustic diffuse field, Eq. (1), is proportional to

$$F(k) = \begin{cases} 1, & |k| \leq k_o \\ 0, & |k| > k_o \end{cases},$$

(12)

where $k$ is the spatial wave number and $k_o$ is $\omega/c_0$, as above. This wave number spectrum is also plotted in Fig. 5, where $k/k_o = i/2N_\lambda$, $i$ is the index for the eigenvalues, and $N_\lambda$ is the number of wavelengths in the analysis window, which is 10 in this case. It can be seen that the theoretical prediction of the eigenvalue distribution, calculated from the wave number spectrum of an infinite array, is reasonably good, and predicts the fall-off in value after the 20th eigenvalue.

Figure 6 shows, in dB, the ratio of the mean square error involved in taking a reduced-rank approximation to the $S_{dd}$ matrix for the simulation of the diffuse acoustic field to the mean square value of the original field, Eq. (11), as a function of the number of uncorrelated components being considered, divided by the number of wavelengths over the array. It is clear that a 10 dB reduction in the mean square error can be obtained with about two uncorrelated components per wavelength. The solid line in Fig. 6 is the theoretical prediction of the error for an infinite array, obtained by integrating the spatial Fourier transform above over a finite range of $k$, to give $J_e$ and over all $k$, to give $J_d$, so that

$$\frac{J_e(N_U)}{J_d} = \begin{cases} 1 - \frac{N_U}{2N_\lambda}, & N_U \leq 2N_\lambda \\ 0, & N_U > 2N_\lambda \end{cases},$$

(13)

where $N_U$ is the number of uncorrelated components and $N_\lambda$ is the number of wavelengths in the analysis window. This slightly underpredicts the actual error, because of the finite size of the array, but is clearly a good guide to the number of uncorrelated components required. The modeling error when the diffuse pressure field is simulated with an array of loudspeakers is also shown in Fig. 6, as will be discussed in Sec. IV.

Similar simulations for a one-dimensional microphone array of 100 microphones over 10 correlation lengths of a TBL pressure field in the spanwise direction give the eigenvalue distribution shown in Fig. 7. Also plotted is the theoretical prediction for these eigenvalues, obtained from the spatial Fourier transform of the infinite spatial correlation function in the $x$ direction, from Eq. (2), which is proportional to

$$F(k_x) = \frac{1}{1 + (k_x L_x)^2},$$

(14)

where $L_x$ is the spanwise correlation length and in this case $k_x L_x = 2\pi i/10$. Maury et al. have also derived the wave number spectrum for a finite-size array and show that the array must cover at least six correlation lengths for the result to approach that of an infinite array. Figure 8 shows the normalized mean square error in taking a reduced-rank approximation to $S_{dd}$ for the spanwise TBL pressure field, plot-
ted against the number of uncorrelated components per correlation length. Also plotted is the theoretical prediction of the normalized error for an infinite array, obtained by integrating $F(k_x)$ over a finite range of $k_x$, to give

$$\frac{J_e(N_U)}{J_d} = 1 - \frac{2}{\pi} \tan^{-1} \left[ \frac{2\pi N_U}{N_c} \right],$$

where $N_U$ is the number of uncorrelated components and $N_c$...
is the number of correlation lengths within the window. This theory predicts that about one uncorrelated component per correlation length is required to obtain a 10 dB reduction in the mean square error, which is close to the simulation results.

Finally, the solid line in Fig. 9 shows the theoretical prediction for the eigenvalue spectrum obtained by taking the Fourier transform of the spatial correlation of the TBL pressure field in the streamwise direction, in Eq. (247), which is proportional to

\[ F(k_y) = \frac{1}{1 + (k_y L_y - \alpha_y)^2}, \]

where \( L_y \) is the correlation length and \( \alpha_y \) is the constant in Eq. (4), and in this case \( k_y L_y = 2\pi i / 10 \). The eigenvalues from the simulation cannot be directly compared with this theory because when they are calculated numerically, they are generally ordered from largest to smallest. Instead the eigenvalues of the asymptotically equivalent circulant matrix have been calculated, and are plotted in Fig. 9, which are again in reasonable agreement with the theoretical prediction. By integrating \( F(k_y) \) over a finite range of \( k_y \), the theoretical form for the normalized mean square error involved in approximating \( S_{dd} \) for a large array of sensors in the streamwise direction is obtained as

\[
\frac{J_d(N_U)}{J_d} = \frac{1 - \frac{2}{\pi} \tan^{-1} \left[ \frac{2 \pi N_U}{N_c \alpha_y} \right]}{1 + \frac{2}{\pi} \tan^{-1} \left[ \alpha_y \right]}. \quad (17)
\]

This is plotted in Fig. 10, together with the results of the numerical simulation over ten correlation lengths. It can be seen that the theory predicts that about two uncorrelated components per correlation length are required to obtain a 10 dB reduction in the mean square error, and this is in reasonable agreement with the simulations. Thus, about twice as many uncorrelated components per correlation length are required to reproduce the TBL pressure field in the streamwise direction compared with the spanwise direction. This is because of the presence of the convective peak in the wave number spectrum, as seen in Fig. 9.

IV. SIMULATION OF THE PRESSURE FIELD WITH ACOUSTIC SOURCES

Having presented a method for determining the number of uncorrelated components necessary to represent a random field at a number of sensors, in this section we will examine how such a random field can be reproduced using a finite number of acoustic sources. The number of uncorrelated components sets a lower bound on the number of acoustic sources required.

The vector of \( N_S \) output spectra from the sensor array, when excited by the set of acoustic sources shown on the right-hand part of Fig. 3, can be written as

\[ y = GWx. \quad (18) \]

where \( x \) is the vector of \( N_R \) reference signal spectra used to generate the inputs to the \( N_A \) actuators via the matrix of filter frequency responses denoted \( W \), and \( G \) represents the matrix of physical frequency responses from the \( N_A \) sources to the \( N_S \) sensors, including the acoustic propagation effects. It will be assumed that the reference signals are white, have unit variance, and are mutually uncorrelated, in which case their spectral density matrix is equal to the identity matrix.
The spectral density matrix of the microphone outputs when subject to the sensor array can thus be written as
\[ S_{yy} = GW \Lambda W^H. \]  

Figure 4 shows the block diagram, used to design the responses of the filter matrix, \( W \). It has been assumed that the set of desired signals at the sensors, \( d \), has been generated by passing the same set of independent white reference signals discussed above, \( x \), through the filter matrix \( P \). Clearly the waveforms of the random pressures in the target pressure field are unknown, but if the spectral density matrix of the sensor outputs, \( S_{dd} \), can be made equal to that of the target pressure field, then the pressure fields are statistically equivalent.

In order to derive the properties of the filter matrix \( P \) in Fig. 4, we again consider the eigenvalue/eigenvector decomposition of the spectral density matrix for the measured pressures when exposed to the desired pressure field;
\[ S_{dd} = \Lambda Q Q^H, \]  
where \( Q \) is the matrix of eigenvectors and \( \Lambda \) the diagonal matrix of real, positive eigenvalues. If the vector of target pressures is generated from \( x \), as in Fig. 4, then
\[ d = Px, \]  
so that
\[ S_{dd} = P \Lambda P^H, \]  
where Eq. (19) for \( S_{xx} \) has again been used.

Since \( \Lambda^{1/2} = (\Lambda^{1/2})^H \), then Eq. (21) can be written as
\[ S_{dd} = (QA^{1/2})(QA^{1/2})^H, \]  
and the statistical properties of the signals, \( d \), generated in Fig. 4 are identical to those of the target pressure field if the number of reference signals is equal to the number of sensors, \( N_R = N_S \), so that \( P \) is a square matrix, and \( P \) is given by
\[ \mathbf{P} = QA^{1/2}. \]  
In principle, the number of reference signals could be reduced to the number of uncorrelated components, in which case \( P \) could be calculated from the eigenvalue/eigenvector decomposition of \( S_{dd} \) in Eq. (8), but we are interested below in determining the minimum number of sources independently of the minimum number of correlated components, and so we retain the full number of reference signals. The matrix of filter responses, \( W \), is now calculated which minimizes the mean square difference between these desired signals and the sensor outputs due to the acoustic sources in Fig. 3. The vector of error signals, defined as the difference between the desired output signals, \( d \), and the generated output signals, \( y \), in Fig. 4, can be written as
\[ e = (P - GW)x. \]  
If these error signals can be made small, then the pressure signals generated by the actuator array must be similar to those of the target pressure field. The magnitude of these error signals can be quantified using a norm of their spectral density matrix, which, since \( S_{xx} = I \), can be written as
\[ S_{ee} = GWW^H G^H - GWW^H PW^H G^H + PP^H. \]
The matrix of control filters $\mathbf{W}$ is now calculated that minimizes the quadratic cost function, $J_e$, equal to the sum of the mean square value of the errors, which is equal to

$$J_e = \text{trace}(\mathbf{S}_{ee}) = \sum_{j=1}^{N} \sum_{j=1}^{N} \mathbf{S}_{jj}.$$  \hfill (28)

It is shown by Elliott\textsuperscript{25} that the complex matrix containing the derivatives of $J_e$ with respect to the real and imaginary parts of each element of $\mathbf{W}$, $\mathbf{W}_g$, and $\mathbf{W}_f$, is equal to

$$\frac{\partial J_e}{\partial \mathbf{W}_g} + j \frac{\partial J_e}{\partial \mathbf{W}_f} = 2 \mathbf{G}^H \mathbf{G} \mathbf{W} - 2 \mathbf{G}^H \mathbf{P}. \hfill (29)$$

Assuming that $\mathbf{G}^H \mathbf{G}$ is nonsingular and setting Eq. (29) to zero allows the optimum least-squares matrix of control filters to be calculated as

$$\mathbf{W}_{opt} = [\mathbf{G}^H \mathbf{G}]^{-1} \mathbf{G}^H \mathbf{P} = \mathbf{G}^{+} \mathbf{P}, \hfill (30)$$

where $\mathbf{G}^{+}$ denotes $[\mathbf{G}^H \mathbf{G}]^{-1} \mathbf{G}^H$, which is the pseudoinverse of the matrix $\mathbf{G}$. The frequency response matrix would probably not be causal, but could be made causal for a real-time implementation by including delays in all elements, which will not affect the statistical properties of the reproduced pressure field. In practice, the drive signals to the actuators could be generated off-line and just played out during synthesis, in which case the filter matrix would not need to be made causal. It is important, however, that the same geometric and physical conditions are maintained during reproduction as were present when the frequency responses contained in $\mathbf{G}$ were measured. The resulting minimum value of the norm of $\mathbf{S}_{ee}$, obtained by substituting Eq. (30) into Eq. (27), is

$$J_{e,\text{min}} = \text{trace}[(\mathbf{I} - \mathbf{G} \mathbf{G}^+) \mathbf{S}_{dd}]. \hfill (31)$$

This can be divided by the trace of $\mathbf{S}_{dd}$, referred to above as $J_d$, to give a normalized value of the minimum mean square error involved in the simulation, which is a useful way of quantifying the degree to which the statistical properties of the target pressure field have been reproduced by the acoustic array.

**V. EXAMPLES OF SIMULATION WITH MONOPOLE ACOUSTIC SOURCES**

One practical actuator arrangement that could be used to simulate a given random pressure field is an array of loudspeakers, which can be modeled as monopole acoustic sources. The number of monopoles required to achieve a given accuracy of simulation will be investigated by again considering a one-dimensional array, as shown in Fig. 11. The simulation of an acoustic diffuse field is considered first.

In this computer simulation there are again assumed to be 100 microphones evenly spaced over a length scale, which includes 10 acoustic wavelengths, but now $N_A$ loudspeakers are evenly spaced at a height $h$ above the surface on which the microphones are positioned. The loudspeaker array is assumed to be driven by the outputs of a matrix of control filters, whose responses are adjusted to minimize the sum of the mean-square differences between the 100 microphone outputs and those due to a corresponding diffuse field excitation, as discussed in Sec. III. A series of simulations has been performed in which the loudspeakers have been modeled as acoustic monopole sources radiating over an infinite rigid surface on which the microphone array has been mounted, and their number, $N_A$, and their height above the microphone array, $h$, has been varied.

Figure 12 shows the variation of the normalized residual mean-square error as a function of the height of the loudspeaker array, divided by the acoustic wavelength $\lambda$, for different numbers of loudspeakers in the array, $N_A$. With $N_A = 16$, the mean-square error falls with height, but little reduction in mean-square error is achieved. When a larger number of loudspeakers is used, $N_A = 24$, which can achieve a more appreciable reduction in mean-square error, these reductions are relatively independent of the height of the loudspeaker array, provided it is greater than about 0.3$\lambda$, but have a minimum when $h = 0.4\lambda$. The control filters are clearly able to compensate for the matrix of acoustic responses between the loudspeakers and microphones, provided the loudspeakers are not too close to the microphones, in which case some microphone outputs are overwhelmingly dominated by the near-field responses of the adjacent loudspeakers. There are numerical conditioning problems if the loudspeakers are too far from the microphones, however, and a reasonable height, $h$, appears to be about half the separation distance between the loudspeakers.

Similar graphs are obtained when simulations are performed to simulate the random TBL pressure field along a one-dimensional array in either the spanwise or streamwise directions.\textsuperscript{22} Although the number of loudspeakers required to achieve a given reduction in the error changes in these cases, near-optimal reductions are achieved when the distance of the sources from the microphone array is again about one-half the separation distance between the loudspeakers. The magnitude of the mean square error reduction, for this separation distance from the monopoles to the microphones, has been plotted as a function of the number of monopoles in the acoustic actuator arrays for the acoustic diffuse in Fig. 6 and for the TBL pressure fields in the span and streamwise directions in Figs. 8 and 10, respectively.

It is interesting to note that the normalized mean square error involved in simulating any of these one-dimensional random pressure fields with an array of suitably positioned monopole sources is only slightly larger than that when the reduced rank approximation to $\mathbf{S}_{dd}$ is considered with an equal number of uncorrelated components. The number of sources in a one-dimensional array of acoustic monopoles thus appears to approach the minimum possible number of uncorrelated sources required. The quality of the synthesis can also be assessed by plotting the correlation functions.
resulting from simulations with various numbers of monopoles per wavelength for the acoustic diffuse field, in Fig. 13, and with various numbers of monopoles per correlation length for the TBL pressure field in the spanwise and streamwise directions, in Figs. 14 and 15, respectively. The spatial correlation structures are reasonably well reproduced when the normalized mean square error is below $-10$ dB, although the peak value for the reproduced TBL pressure field still does not reproduce the sharp cusp of the Corcos model, which may not be physically realistic in any case, as discussed in Sec. II.

Although the current theory allows clear conclusions to be drawn about the number of sources required to simulate diffuse or TBL pressure fluctuations along a one-dimensional

![Graph showing normalized mean-square error as a function of normalized distance of the sources from the microphone array, $h/l_0$.]

**FIG. 12.** The normalized mean-square error for simulations in which various numbers of acoustic sources, $N_A$, were used to reproduce a diffuse pressure field, as a function of the normalized distance of the sources from the microphone array, $h/l_0$.

**FIG. 13.** The target spatial correlation function of the acoustic diffuse field (solid line), and that of the approximate pressure field generated using 1.8 monopoles per acoustic wavelength (dotted line), 2 monopoles per acoustic wavelength (dashed line), and 2.2 monopoles per acoustic wavelength (thin solid line), plotted as a function of normalized separation distance from the center microphone, divided by the acoustic wavelength. The numbers in brackets are the changes in mean square error associated with each number of monopoles.
sensor array, the theoretical results do not generalize easily to two-dimensional arrays. It is still possible, however, to use the theory outlined in Sec. III to calculate the best least-squares approximation to a given random pressure field over a two-dimensional array of microphones. Figure 16, for example, shows the results of a simulation in which a 5×5 array of monopoles has been used to generate a least squares approximation to a diffuse or a TBL pressure field over a 100×100 array of microphones on the surface of the 0.7 m×0.5 m×0.003 m panel, as in the simulation used to generate in Fig. 2. The overall kinetic energy of the panel is again plotted as a function of frequency, with that generated by the pressure field synthesized with the monopole array shown dotted, and that due to the theoretical pressure fields, whose spatial correlation functions are given by Eqs. (1) and (2), shown solid.

It can be seen that the panel’s response to the diffuse pressure field is reproduced reasonably well by the monopole array up to about 800 Hz. At this frequency the largest spacing between the monopoles is about one-third of the acoustic wavelength, in reasonable agreement with the maximum separation of one-half wavelength predicted by the one-dimensional theory. In current sound transmission suites, the Schröder frequency of the reverberant source room is typically well below 800 Hz. These simulations suggest that a much smaller and more heavily damped source room could be employed in sound transmission suites if a near-field loudspeaker array was used to reproduce the diffuse incident pressure field up to several hundred Hertz, and the naturally diffuse nature of the reverberant sound field in the source room was used to generate the incident pressure field only above this frequency.

The panel’s response to the simulated TBL excitation is very similar to that of the ideal theoretical model only up to about 200 Hz, however. At this frequency the separation distance between the monopoles is about four-fifths of a correlation length in the spanwise direction, whereas the one-dimensional theory above would suggest that the maximum separation would be equal to the correlation length. Once again the one-dimensional theory provides a reasonable guide to the number of acoustic sources required, although the small spanwise correlation lengths severely limit the frequency range over which a 5×5 array can adequately reproduce the TBL pressure field.

VI. INITIAL EXPERIMENTAL RESULTS

As a preliminary investigation of the practical problems associated with simulating various random pressure fields with an array of loudspeakers, an enclosure was constructed that housed 16 loudspeakers and 32 microphones, as illustrated in Fig. 17. The top of the enclosure had internal dimensions of approximately 430×430 mm. The 74 mm diameter loudspeakers were mounted in a four by four array so that their cones were about 40 mm below a 1 mm thick aluminum panel, which could be mounted on top of the enclosure, but is not shown in Fig. 17, on whose lower surface the pressure field was to be generated. About 10 mm below the panel, a grid of brass rods was mounted, onto which were attached 32 miniature electret microphones. The frequency response from each loudspeaker to each microphone was measured up to about 1 kHz, with the panel in place, to form the experimental version of the matrix G in Eq. (18), both with and without acoustic damping material in

![Graph showing target spatial correlation function of TBL pressure field in the spanwise direction](image_url)

**FIG. 14.** The target spatial correlation function of the TBL pressure field in the spanwise direction (solid line), and that of the approximate pressure field generated using 0.5 monopoles per correlation length (dotted line), 1 monopole per correlation length (dashed line), and 1.5 monopoles per correlation length (thin solid line), plotted as a function of the normalized separation distance from the center microphone, divided by the correlation length. The numbers in brackets are the changes in mean square error associated with each number of monopoles.
the enclosure and mechanical damping material on the plate. The ratio of the predicted minimum mean square error when simulating various random sound fields at the 32 microphones, as specified by the matrix $S_{dd}$, was then calculated from Eq. (31) and divided by the trace of $S_{dd}$ to give the normalized error, as above. For the undamped enclosure the normalized error increased significantly at frequencies corresponding to the coupled natural frequencies of the panel and acoustic enclosure. At these frequencies it is very difficult for the loudspeakers to generate anything but the pressure field corresponding to modal response. When the system is more heavily damped, these resonances are still apparent in the individual responses of $G$, but they are not evident in the graphs of normalized error, indicating that they have been equalized by the optimum controller in Eq. (30). Figure 18 shows the normalized error calculated from the measured $G$ matrix and Eq. (31) for the damped enclosure when attempting to simulate a perfect acoustic diffuse field (thick line) and a pressure field corresponding to the TBL, whose properties are described in Sec. II (thin line).
It is clear from Fig. 18 that in the case of the diffuse field, the simulation error is relatively low up to about 600 Hz, at which frequency the separation between the loudspeakers is about one-third of an acoustic wavelength, whereas in the case of the TBL pressure field, the simulation error is somewhat higher, and rises significantly at about 200 Hz, at which frequency the loudspeaker separation distance is about four-fifths of the correlation length in the spanwise direction. The experimental results are thus in broad agreement with the simulation results presented in Sec. V, provided lightly damped resonances are avoided.

VII. CONCLUSIONS

The framework presented allows the feasibility to be assessed of generating a pressure field with a given spatial
correlation structure using an array of acoustic actuators. The pressure field is specified in terms of the spectral density matrix of the outputs of an array of sensors, which can be calculated from a theoretical model of the spatial correlation structure of the pressure field to be synthesized, or measured in an experimental application.

The formulation allows a lower bound to be calculated on the number of actuators required to accurately generate a given pressure field, which is equal to the number of uncorrelated components required to approximate the statistical properties of the field to a required degree of accuracy. A one-dimensional simulation suggests that two uncorrelated components per wavelength are required to approximate a diffuse acoustic field. This is in good agreement with the theoretical prediction obtained from a wave number decomposition of the diffuse field, which indicates that exactly two uncorrelated components per wavelength are required for an infinite array. For the Corcos model of the TBL pressure field, this number scales with the correlation length, and for a 10% mean square error in the simulation, about one uncorrelated component per correlation length is required in the streamwise direction and about two per correlation length are required in the spanwise direction. Although a mean-square error criterion has been used here, other error criteria could be used in this formulation, emphasizing different aspects of the spatial correlation structure. The level of error criterion that is acceptable will depend on the use to which the simulation is to be put. If the target pressure field is calculated from a model, the accuracy of this model must also be taken into account when assessing the physical significance of the error criterion.

The framework has been used to assess the feasibility of synthesizing a diffuse sound field with an array of acoustic sources close to a surface. Such an arrangement may be useful to replace the source room in sound transmission tests, or sources close to a surface. Such an arrangement may be used to assess the physical significance of the error criterion.

FIG. 18. Normalized error as a function of frequency calculated from the $G$ matrix measured in the experimental enclosure when simulating an acoustic diffuse field (dark curve) and the pressure field due to a TBL (light curve).

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16. B. M. Elliott, “Characteristics of the field of turbulent wall pressure

most the same as the number of uncorrelated components in the pressure field.

When simulations are performed to reproduce TBL pressure fields with acoustic monopoles, the number of monopoles required to reproduce the pressure field is again approximately equal to the number of uncorrelated components. The simple Corcos model predicts that the correlation length is inversely proportional to frequency, and so the number of sources required to reproduce such a TBL model rises rapidly with frequency. Simple calculations suggest that the number of loudspeakers required to synthesize pressure fields due to a TBL at high subsonic Mach numbers may be technologically feasible up to a few hundred Hertz. A number of practical issues need to be addressed before such a simulation technique can be used with any confidence in practice, but a preliminary experimental study provides results that are in broad agreement with the simulations, provided lightly damped resonances in the driving acoustic field are avoided.