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par

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Sujet : *Analyse théorique et numérique de la dynamique des densités de dislocations*

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Résumé

Ce travail porte sur l'analyse théorique et numérique de la dynamique des densités de dislocations. Les dislocations sont des défauts qui se déplacent dans les cristaux, lorsque ceux-ci sont soumis à des contraintes extérieures. D'une façon générale, la dynamique des densités de dislocations est décrite par un système d'équations de transport, où les champs de vitesse dépendent de manière non-locale des densités de dislocations.

Dans une première partie, nous nous plaçons dans un cadre unidimensionnel. Nous démontrons pour un système 2×2 simplifié des résultats d'existence globale et d'unicité de solution, ainsi qu'une estimation d'erreur entre la solution continue et son approximation numérique en utilisant un schéma aux différences finies. Puis, en se basant sur une nouvelle méthode d'estimation de l'entropie du gradient, nous démontrons un résultat d'existence globale et quelques résultats d'unicité pour des systèmes hyperboliques diagonalisables en dimension 1.

Dans une seconde partie, nous nous intéressons à un cadre plus général de la dynamique des densités de dislocations où nous étudions un modèle bidimensionnel. Ce modèle a été introduit par Groma et Balogh [71]. Nous démontrons dans ce cadre un résultat d'existence globale en mettant en œuvre l'estimation sur l'entropie du gradient des solutions. Des simulations numériques de ce modèle sont aussi présentées.

Abstract

This thesis deals with theoretical and numerical analysis of the dynamics of dislocation densities. Dislocations are the moving defects in the crystals, under the presence of an exterior stress. More generally, the dynamics of dislocation densities is described by a system of transport equations where the velocity field depends non locally on the dislocation densities.

In the first part, we consider a one dimensional framework. We prove, for a simplified 2×2 system, some global existence and uniqueness results of solutions as well as an error estimate between the continous solutions and its numerical approximation, by using a finite difference scheme. Then, based on a new gradient entropy estimate method, we prove a global existence and some uniqueness results for some one dimensional diagonalizable hyperbolic systems.

In the second part of the thesis, we are intersted in a more general framework of dislocation densities, where we study a bidimensional model, introduced by Groma and Balogh [71]. For this model we prove a global existence result via the new gradient entropy estimate of the solutions. Some numerical simulations for this model are presented as well.

Publications issues de la thèse

Articles acceptés

- *Well-Posedness Theory for a Nonconservative Burgers-Type System Arising in Dislocation Dynamics*, SIAM Journal on Mathematical Analysis, 39 (2007), pp. 965-986. (cf. chapitre 2)

- (avec N. Forcadel), *A convergent scheme for a non-local coupled system modelling the dynamics of dislocation densities*, à paraître dans Mathematics of Computation. (cf. chapitre 3)

Articles soumis

- (avec M. Cannone, R. Monneau et F. Ribaud), *Global existence for a system of non-linear and non-local transport equations describing the dynamics of dislocation densities*, soumis à Archive for Rational Mechanics and Analysis. (cf. chapitre 5)

Article en fin de rédaction

- (avec R. Monneau) *Global continuous solutions to diagonalizable hyperbolic systems with large and monotone data.* (cf. chapitre 4)

Sommaire

Introduction générale

Cette thèse porte sur l'étude mathématique de la dynamique des densités de dislocations. Cette dynamique est modélisée par des systèmes d'équations de transport non-linéaires et non-locales. Nous nous intéressons à l'analyse de ces systèmes d'équations ainsi qu'à leurs résolutions numériques.

Cette introduction se compose de trois sections. Dans la première section, nous présentons un type de modèle bidimensionnel qui sera étudié dans ce mémoire. Dans la seconde section, nous nous intéressons à un sous-modèle unidimensionnel, et nous exposons les résultats obtenus à l'issu de ce travail. Dans la troisième section, nous reviendrons à l'étude du modèle bidimensionnel et nous présenterons notre résultat.

Les hypothèses utilisées dans les théorèmes qui suivent ne sont pas optimales. Nous avons fait ce choix dans un souci de simplification de la présentation. Nous renvoyons aux chapitres de la thèse correspondants pour des résutats plus généraux.

1 Type de modèles étudiés dans cette thèse

1.1 Motivations physiques

Une dislocation est une ligne de défaut qu'on trouve dans les cristaux réels. Par exemple, dans la Figure 1, les lignes noires observées sont les dislocations présentes dans un alliage (Ni_3Al), pris en microscopie électronique, à l'échelle $1\mu m = 1^{-9}m$.

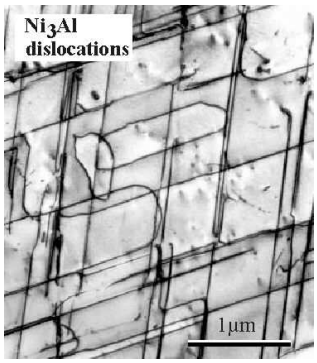

FIG. 1 – Observation de dislocations au micro électronique

Ces défauts ont été introduits dans les années trente par G. I. Taylor [128], E. Orowan [115] et M. Polanyi [119], comme une explication principale à l'échelle microscopique des déformations plastiques des matériaux. Sous l'effet de contraintes extérieures, ces dislocations peuvent se déplacer dans un plan cristallographique bien défini appelé "plan de glissement". Le déplacement d'une dislocation est caractérisé par un vecteur $\vec{b}$ appelé "vecteur de Burgers" [23].

La quantité de dislocations dans un cristal est représentée par sa densité, définie par le nombre de lignes de dislocations qui traversent une section unitaire. Nous nous sommes interéssés, dans cette thèse, à l'étude de la dynamique des densités de dislocations. Notre motivation pour cette étude est de mieux comprendre le comportement mécanique issu de la déformation plastique des matériaux, en présence de forte concentration de dislocations.

Plus précisément, nous avons considéré des modèles particuliers qui décrivent la dynamique des densités de dislocations. Dans un premier temps, nous nous sommes consacrés aux propriétés qualitatives (existence, unicité,...) d'un système unidimensionnel simplifié. Puis nous avons considéré le cadre plus général, donné par le modèle bidimensionnel introduit par I. Groma et P. Balogh [71].

1.2 Présentation d'un modèle bidimensionnel

Dans cette sous-section, nous présentons de manière simple le modèle de Groma-Balogh introduit dans [71] et nous donnons les équations relatives à ce modèle, qui seront étudiées par la suite. Nous renvoyons à la section 3 du chapitre 1 pour une dérivation détaillée des équations présentées ici.

La dynamique des densités de dislocations relève d'un problème tridimensionnel. Néanmoins dans le cas particulier où les lignes de dislocations sont des droites paralléles dans l'espace tridimensionnel, elles peuvent être vues comme des points dans un plan transversal. Dans le cadre où les dislocations se déplacent suivant une seule direction fixe, donnée par un vecteur de Burgers $\vec{b} = (1, 0)$, I. Groma et P. Balogh [71] ont décrit une évolution continue bidimensionnelle des densités de dislocations. Dans ce modèle, nous considérons deux types de dislocations dans le plan (x_1, x_2). Pour un champ de vitesse donné, les dislocations de type $(+)$ se propagent suivant la direction du vecteur $+\vec{b}$, et celles de type $(-)$ se propagent suivant la direction $-\vec{b}$ (voir Figure 2).

Ce modèle est régi par un système d'équations de transport où le champ de vitesse est la contrainte de cisaillement dans le cristal. Ce champ de vitesse est résolu à partir de l'équation de l'élasticité linéaire, qui peut être écrite comme une quantité non-locale

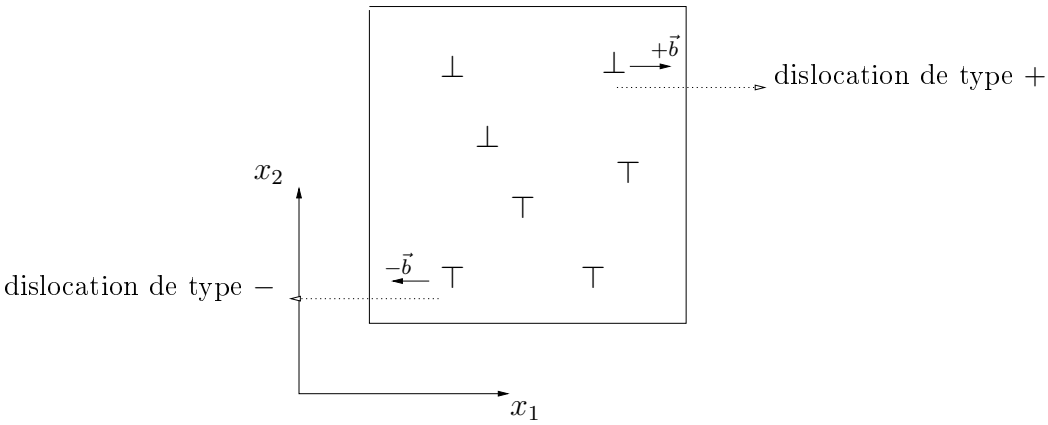

FIG. 2 – Modèle 2D de Groma-Balogh.

dépendant des densités de dislocations. Plus précisément, le système d'évolution est donné par :

$$\begin{cases} \dfrac{\partial \rho^+}{\partial t}(x_1,x_2,t) = -\left(c_0 * \left(\rho^+(\cdot,t) - \rho^-(\cdot,t)\right)(x_1,x_2)\right)\dfrac{\partial \rho^+}{\partial x_1}(x_1,x_2,t) & \text{sur } \mathbb{R}^2 \times (0,T), \\ \dfrac{\partial \rho^-}{\partial t}(x_1,x_2,t) = \left(c_0 * \left(\rho^+(\cdot,t) - \rho^-(\cdot,t)\right)(x_1,x_2)\right)\dfrac{\partial \rho^-}{\partial x_1}(x_1,x_2,t) & \text{sur } \mathbb{R}^2 \times (0,T). \end{cases} \tag{1.1}$$

Les inconnues du système (1.1), sont les scalaires ρ^+ et ρ^- à l'instant t et à la position (x_1, x_2) qu'on notera pour simplifier $\rho^\pm$. Ces derniers correspondent aux déformations plastiques du matériau. Leurs dérivées par rapport à x_1, $\dfrac{\partial \rho^\pm}{\partial x_1}$ représentent les densités de dislocations de type $\pm$. La fonction $c_0(x_1, x_2)$ est un noyau donné dépendant des coefficients élastiques du matériau. Ce noyau sera défini explicitement dans les sections suivantes. Ici, la convolution a lieu en espace uniquement, le terme de convolution $c_0 * (\rho^+ - \rho^-)$ représente la force créée par le champ élastique généré par les lignes de dislocations.

1.3 Guide de lecture de l'introduction

Dans une première partie, nous allons nous intéresser à l'étude d'un modèle unidimensionnel simplifié. Plus précisément, nous présentons un système (2×2) unidimensionnel qui est un sous-modèle de (1.1) pour lequel nous exposons, dans un premier temps, un résultat d'existence et d'unicité dans l'espace de Sobolev, et

dans un second temps, un résultat d'existence et d'unicité d'une solution de viscosité ainsi qu'une estimation d'erreur de type Crandall-Lions entre la solution continue et la solution discrète. Puis, nous finirons cette étude unidimensionnelle en annonçant un théorème d'existence globale et quelques résultats d'unicité pour un système hyperbolique diagonalisable en dimension 1.
Dans une seconde partie, nous donnons un résultat d'existence globale dans un cadre bidimensionnel général.

2 Première partie : modèle unidimensionnel

Nous remarquons que la difficulté principale dans le cas unidimensionnel est de travailler avec des systèmes d'équations fortement couplées. Cependant, la présence du terme non-local ne pose pas de difficultés dans l'étude unidimensionnelle contrairement au cas bidimensionnel.

2.1 Résultat d'existence et d'unicité globale dans $H^1_{loc}(\mathbb{R} \times [0,T))$

Nous nous intéressons maintenant à un sous-modèle unidimensionnel du système (1.1), pour lequel nous allons montrer que le problème est bien posé.

De manière plus précise, dans ce sous-modèle, nous supposons que les densités de dislocations dépendent d'une seule variable $x = x_1 + x_2$, où (x_1, x_2) sont les coordonnées d'un point de $\mathbb{R}^2$ (voir Figure 3) et nous considèrons le problème sur le domaine 1-périodique en x. Ce cadre particulier permet de ramener le système bidimensionnel (1.1) à un système unidimensionnel (nous renvoyons à la section 3 du chapitre 3 pour une modélisation physique).

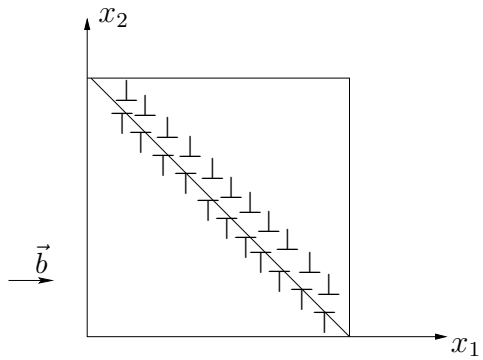

FIG. 3 – Sous-modèle 1D invariant par translation dans la direction (-1,1).

Ce sous-modèle $1D$ est décrit par un système d'équations de transport couplées non-linéaires et non-locales :

$$\begin{cases} \dfrac{\partial \rho^+}{\partial t}(x,t) &= -\left((\rho^+ - \rho^-)(x,t) + \alpha \displaystyle\int_0^1 (\rho^+ - \rho^-)(y,t)dy\right) \dfrac{\partial \rho^+}{\partial x}(x,t) \quad \text{sur} \quad \mathbb{R} \times (0,T), \\ \dfrac{\partial \rho^-}{\partial t}(x,t) &= \left((\rho^+ - \rho^-)(x,t) + \alpha \displaystyle\int_0^1 (\rho^+ - \rho^-)(y,t)dy\right) \dfrac{\partial \rho^-}{\partial x}(x,t) \quad \text{sur} \quad \mathbb{R} \times (0,T). \end{cases} \tag{2.2}$$

Comme indiqué précédemment, les inconnues sont les scalaires ρ^+, ρ^-. Ici la constante α est donnée et dépend des coefficients élastiques et de la taille du matériau. La présence du terme non-local, $\alpha \displaystyle\int_0^1 (\rho^+ - \rho^-)(y,t)dy$, est dûe à l'intégration de l'équation d'élasticité sur la boîte $[0,1)$ périodique.

Ce système est complété par les conditions initiales suivantes :

$$\rho^\pm(x, t=0) = \rho_0^\pm(x) = \rho_0^{\pm,per}(x) + L_0 x \quad \text{sur } \mathbb{R} \tag{2.3}$$

où $\rho_0^{\pm,per}(x)$ sont des fonctions 1-périodiques. Nous modèlisons une distribution périodique de dislocations $+$ et $-$ avec une densité totale L_0 de chaque espèce par période spatiale de longueur 1. La périodicité est une façon de déduire ce qui se passe à l'intérieur du matériau loin de ses bords.

Le premier résultat concernant ce système est un résultat d'existence et d'unicité dans $H^1_{loc}(\mathbb{R} \times (0,T))$ où pour tout Ω ouvert de $\mathbb{R}^N$, $N = 1, 2$, nous définissons

$$H^1_{loc}(\Omega) = \{f \text{ telle que } f \in L^2_{loc}(\Omega) \text{ et } \nabla f \in L^2_{loc}(\Omega)\}.$$

Théorème 2.1 (Existence et unicité pour le système (2.2)-(2.3), *[47, Théorème 1.1])*
Pour tous $T, L_0 \geq 0$, $\alpha \in \mathbb{R}$ et pour toutes données initiales $\rho_0^\pm \in H^1_{loc}(\mathbb{R})$ et sous les hypothèses suivantes :

(H1) $\rho_0^\pm(x+1) = \rho_0^\pm(x) + L_0$, *(fonction 1-périodique + fonction linéaire),*

(H2) $\dfrac{\partial \rho_0^\pm}{\partial x} \geq 0$, *presque partout dans $\mathbb{R}$, ($\rho_0^\pm$ croissante),*

le système (2.2)-(2.3) admet une unique solution $\rho^\pm \in H^1_{loc}(\mathbb{R} \times [0,T))$, au sens des distributions, telle que, pour presque tout $t \in (0,T)$, la fonction $\rho^\pm(.,t) : x \longmapsto \rho^\pm(x,t)$ vérifie (H1) et (H2).

La preuve de ce théorème est décomposée de la manière suivante : tout d'abord, nous régularisons le système (2.2) par l'addition d'un terme de viscosité $\varepsilon \dfrac{\partial^2 \rho^{\pm,\varepsilon}}{\partial x^2}$, puis nous montrons sur la solution $\rho^{\pm,\varepsilon}$ du système régularisé une estimation *a priori* uniforme en ε dans $L^\infty((0,T); H^1_{loc}(\mathbb{R}))$. Plus précisément, l'estimation clé est une estimation d'énergie sur les densités de dislocations sur le tore $\mathbb{T} = \mathbb{R}/\mathbb{Z}$:

$$\left\| \frac{\partial}{\partial x}(\rho^{+,\varepsilon} - \rho^{-,\varepsilon}) \right\|^2_{L^\infty((0,T);L^2(\mathbb{T}))} + \left\| \frac{\partial}{\partial x}(\rho^{+,\varepsilon} + \rho^{-,\varepsilon}) \right\|^2_{L^\infty((0,T);L^2(\mathbb{T}))} \leq B_0, \tag{2.4}$$

où $B_0 = \left\| \dfrac{\partial}{\partial x}(\rho_0^+ - \rho_0^-) \right\|^2_{L^2(\mathbb{T})} + \left\| \dfrac{\partial}{\partial x}(\rho_0^+ + \rho_0^-) \right\|^2_{L^2(\mathbb{T})}$.

Cette estimation mène à un résultat d'existence globale et permet le passage à la limite quand ε tend vers 0 en utilisant un argument de compacité. Finalement, la démonstration de l'unicité est faite d'une manière directe en utilisant une contraction dans $L^\infty((0,T); L^2_{loc}(\mathbb{R}))$.

Nous mentionnons que le système (2.2) est lié à d'autres modèles bien connus, comme les équations de transport avec des champs de vecteurs peu réguliers. Ces équations ont été étudiées par R. J. DiPerna, P. L. Lions dans [45], où les auteurs ont montré l'existence et l'unicité de solution (au sens de solutions renormalisées), pour des champs de vecteurs $L^1((0,T); W^{1,1}_{loc}(\mathbb{R}))$ à divergence $L^1((0,T); L^\infty(\mathbb{R}))$. Puis cette étude a été généralisée par L. Ambrosio [8], en considérant des champs $L^1((0,T); BV_{loc}(\mathbb{R}))$ à divergence bornée. Ici, nous travaillons dans la dimension $N = 1$ et nous montrons l'existence et l'unicité d'une solution du système (2.2)-(2.3) avec un champ de vecteur (c'est-à-dire la vitesse) seulement dans $L^\infty((0,T), H^1_{loc}(\mathbb{R}))$. Dans notre étude, la divergence des champs de vecteurs est non bornée. Mais nous avons pu aboutir à ce résultat en contrôlant le gradient de la solution dans $L^\infty((0,T), H^1_{loc}(\mathbb{R}))$.

Il y a aussi plusieurs travaux qui ont été faits dans le cas des systèmes strictement hyperboliques de la forme :

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + F(u)\dfrac{\partial u}{\partial x}(x,t) = 0 & u(x,t) \in U,\ x \in \mathbb{R},\ t \in (0,T), \\ \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{2.5}$$

où l'espace des états U est un sous-ensemble ouvert de $\mathbb{R}^M$, et F une fonction de classe C^1 de U dans l'ensemble des matrices carrées d'ordre M. De plus, les auteurs ont supposé que $F(u)$ possède M valeurs propres réelles distinctes notées :

$$a^1(u) < a^2(u) < ... < a^M(u),$$

On remarque que cette condition sur les valeurs propres n'entre pas dans le cadre de ce travail même dans le cas où $\alpha = 0$, parce que nous n'avons pas une propriété de signe sur $\rho^+ - \rho^-$.

Nous citons ici quelques résultats bien connus pour les systèmes 2×2 strictement hyperboliques et nous renvoyons à la sous-section 2.3 pour des références concernant le cadre général ($M \times M$). P. D. Lax [98] a prouvé l'existence et l'unicité des solutions régulières croissantes des systèmes 2×2 strictement hyperboliques. Le résultat de Lax a été également prouvé par D. Serre [123, vol. II] pour des systèmes $M \times M$ hyperboliques riches. D. Serre [122] a étudié le cas des systèmes 2×2 de Temple où il a démontré l'existence globale d'une solution à variation totale bornée.

Également, dans le cas des systèmes (2×2) strictement hyperboliques R. J. DiPerna [43, 44] a montré l'existence de solutions L^∞. La démonstration de DiPerna est reliée à un argument de compacité par compensation, basée sur la représentation de la limite faible en termes de mesure de Young, ce qui doit se réduire à une masse de Dirac en raison de la présence d'une famille nombreuse d'entropies. Ce résultat est basé sur l'idée de L. Tartar [127].

Quand le système est hyperbolique et symétrique, ce qui correspond au cas $\alpha = 0$ dans notre système (2.2), on trouve dans D. Serre [123, Vol I, Th 3.6.1] un résultat d'existence et d'unicité locale dans $C([0,T); H^s(\mathbb{R}^N)) \cap C^1([0,T); H^{s-1}(\mathbb{R}^N))$, avec $s > \frac{N}{2} + 1$, ce résultat reste local en temps, même dans la dimension $N = 1$.

Une autre propriété intéressante à signaler pour ce système est le principe de comparaison dans le cas $\alpha = 0$.

Théorème 2.2 (Principe de comparaison pour (2.2) avec $\alpha = 0$, *[47, Théorème 1.2]*)
Soient $\rho_1^{\pm,}$, $\rho_2^{\pm} \in H^1_{loc}(\mathbb{R} \times [0,T))$ deux solutions du système (2.2) avec $\alpha = 0$. On suppose que $\rho_1^{\pm}(.,t)$, $\rho_2^{\pm}(.,t)$ vérifient $(H1)$ et $(H2)$, pour presque tout $t \in (0,T)$. Alors,

$$\textit{si } \rho_1^{\pm}(\cdot,0) \leq \rho_2^{\pm}(\cdot,0) \textit{ sur } \mathbb{R}, \textit{ nous avons } \rho_1^{\pm} \leq \rho_2^{\pm} \textit{ presque partout sur } \mathbb{R} \times (0,T).$$

Ce résultat de comparaison est crucial pour un résultat que l'on annoncera par la suite, qui est un résultat d'existence et d'unicité de solution Lipschitz au sens des solutions de viscosité, ainsi que pour une estimation d'erreur entre la solution continue et une approximation numérique. Ici, ce résultat justifie le fait que nous avons pu obtenir l'unicité des solutions.

Remarque 2.3 (Existence et unicité pour l'équation de Burgers)
Nous remarquons que cette technique peut être appliquée au cas d'équation de Burgers classique dans $W^{1,p}_{loc}(\mathbb{R}\times[0,T))$ *pour tout* $1\leq p<+\infty$.

En effet, si nous considérons pour une fonction donnée f *et une donnée initiale* u_0, *l'équation suivante :*

$$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial}{\partial x}(f(u))=0 & \text{sur} \quad \mathbb{R}\times(0,T), \\ \\ u(x,0)=u_0(x) & x\in\mathbb{R}, \end{cases} \tag{2.6}$$

alors nous avons le Théorème suivant :

Théorème 2.4 (Burgers classique, *[47, Théorème 1.4]*)
Soient $T\geq 0$, $p\in[1,+\infty)$ *et* f *localement Lipschitzienne et convexe. Alors, pour toutes données initiales* $u_0\in L^\infty(\mathbb{R})$, *telles que* $\dfrac{\partial u_0}{\partial x}\in L^p(\mathbb{R})$ *et vérifient* $(H2)$, *l'équation (2.6) admet une unique solution* $u\in W^{1,p}_{loc}(\mathbb{R}\times[0,T))$ *qui satisfait* $(H2)$, *pour presque tout* $t\in(0,T)$.

2.2 Résultat d'existence et d'unicité d'une solution de viscosité et une estimation d'erreur discrète-continue

Dans cette sous-section, nous étudions le système (2.2) en utilisant une autre approche qui est la solution de viscosité. Plus prècisément, nous énonçons un résultat d'existence et d'unicité de solutions Lipschitz au sens des solutions de viscosité. Puis, nous proposons un schéma numérique aux différences-finies de type "Upwind" pour lequel nous montrons une estimation d'erreur de type Crandall-Lions [39] entre la solution continue et la solution discrète. Enfin, nous présentons quelques simulations numériques.

2.2.1 Existence et unicité d'une solution de viscosité

Étant donné que les densités de dislocations $\dfrac{\partial\rho^\pm}{\partial x}$ sont des quantités positives, alors nous pouvons voir le système (2.2) comme un système d'équation de Hamilton-Jacobi, en remplaçant $\dfrac{\partial\rho^\pm}{\partial x}$ par $\left|\dfrac{\partial\rho^\pm}{\partial x}\right|$. Ce point de vue nous ramène à l'étude du système d'équations de Hamilton-Jacobi non-locales suivant :

$$\begin{cases} \dfrac{\partial \rho^+}{\partial t} & = -\left((\rho^+ - \rho^-) + \alpha \displaystyle\int_0^1 (\rho^+ - \rho^-)\right) \left|\dfrac{\partial \rho^+}{\partial x}\right| & \text{sur } \mathbb{R} \times (0,T), \\ \dfrac{\partial \rho^-}{\partial t} & = \left((\rho^+ - \rho^-) + \alpha \displaystyle\int_0^1 (\rho^+ - \rho^-)\right) \left|\dfrac{\partial \rho^-}{\partial x}\right| & \text{sur } \mathbb{R} \times (0,T), \\ \rho^\pm(\cdot, t=0) & = \rho_0^\pm(\cdot) & \text{sur } \mathbb{R}. \end{cases} \tag{2.7}$$

où $\rho_0^\pm$ sont définis dans (2.3). Un cadre naturel pour étudier les solutions de ce système est la théorie des solutions de viscosité. Cette notion de solutions de viscosité est assez récente. Elle a été introduite au début des années 1980 par M. G. Crandall et P. L. Lions [38, 39] pour résoudre les équations d'Hamilton-Jacobi du premier ordre. La théorie s'est ensuite étendue aux équations du second ordre pour lesquelles elle a connu un développement spectaculaire après les travaux de R. Jensen [88] et H. Ishii [81]. Pour une bonne introduction à cette théorie, nous renvoyons à G. Barles [12, 13], M. Bardi, I. Capuzzo-Dolcetta [10] et G. Crandall, H. Ishii, P. L. Lions [37].

Cette théorie a été formalisée dans le cadre des systèmes par H. Ishii, S. Koike [83] et H. Ishii [82]. Dans la suite nous rappellerons la définition des solutions de viscosité, proposée par H. Ishii, S. Koike [83]. Cette définition sera utile dans l'annonce de notre résultat. Tout d'abord, nous devons considèrer le problème de Cauchy, pour $i = 1, \ldots, M$:

$$\begin{cases} \partial_t u^i(x,t) + H_i\left(t, u(x,t), \dfrac{\partial u^i}{\partial x}(x,t)\right) = 0 & \text{avec } \ u = (u^i)_i \in \mathbb{R}^M, \ x \in \mathbb{R}, \ t \in (0,T), \\ u^i(x,0) = u_0^i(x) & x \in \mathbb{R}, \end{cases} \tag{2.8}$$

où pour $i = 1, \ldots, M$ $u_0^i \in C(\mathbb{R})$ et $H_i : (0,T) \times \mathbb{R}^M \times \mathbb{R} \longmapsto \mathbb{R}$ sont les Hamiltoniens qui sont supposés continus. On note par

$SCS(\mathbb{R}\times(0,T)) = \{f$ telle que f est une fonction semi-continue supérieurement sur $\mathbb{R} \times (0,T)\}$,

$SCI(\mathbb{R}\times(0,T)) = \{f$ telle que f est une fonction semi-continue inférieurement sur $\mathbb{R} \times (0,T)\}$.

Définition 2.5 (Sous-solution, sur-solution et solution de viscosité pour (2.8))
Une fonction $u = (u^i)_i \in [SCS(\mathbb{R} \times (0,T))]^M$ (resp. $u \in [SCI(\mathbb{R} \times (0,T))]^M$) est

une sous-solution de viscosité (resp. sur-solution) du système (2.8), si pour tout $i = 1, \dots, M$,

- $u^i(\cdot, t = 0) \leq u_0^i$ *(resp.* $u^i(\cdot, t = 0) \geq u_0^i$*),*
- *pour toute fonction test* $\phi \in C^1(\mathbb{R} \times (0, T))$ *telle que* $u^i - \phi$ *atteint un maximum local (resp. minimum local) au point* $(x_0, t_0) \in \mathbb{R} \times (0, T)$*, alors nous avons*

$$\frac{\partial \phi}{\partial t}(x_0, t_0) + H_i(t_0, u(x_0, t_0), \frac{\partial \phi}{\partial x}(x_0, t_0)) \leq 0,$$

$$(\text{resp. } \frac{\partial \phi}{\partial t}(x_0, t_0) + H_i(t_0, u(x_0, t_0), \frac{\partial \phi}{\partial x}(x_0, t_0)) \geq 0).$$

Finalement, u est une solution de viscosité du système (2.8) si et seulement si u est une sous et sur-solution de (2.8).

Théorème 2.6 (Existence et unicité pour le système (2.7)), *[48, Théorème 1.1])*
Soient $T \geq 0$ *et* $L_0, \alpha \in \mathbb{R}$*. On suppose que* $\rho_0^{\pm} \in Lip(\mathbb{R})$*. Alors le système (2.7) admet une unique solution de viscosité* (ρ^+, ρ^-) *uniformément Lipschitz en espace et en temps. Cette solution vérifie la Définition (2.5) quand le terme non-local est gelé. De plus, si au temps initial, nous avons* $\frac{\partial \rho_0^{\pm}}{\partial x} \geq 0$ *presque partout dans* $\mathbb{R}$*, alors ceci reste vrai pour* $t \in (0, T)$*.*

Nous démontrons ce résultat en se basant sur le fait que le système (2.7) vérifie le principe de comparaison au sens de H. Ishii, S. Koike [83] dans le cas $\alpha = 0$. Ceci nous permet d'adapter la démonstration de H. Ishii, S. Koike [83] et d'obtenir l'existence et l'unicité de solution quand le terme non-local est gelé (voir Lemme 4.4 du Chapitre 3 pour une démonstration détaillée). Ensuite, nous utilisons une estimation sur la norme L^∞ du gradient de la solution et nous démontrons par un théorème de point fixe, le résultat pour le système non-local.

Le cadre des données initiales croissantes a été également considéré dans l'étude de l'équation d'Euler pour des fluides compressibles en dimension 1. En ce qui concerne ces études, nous nous référons à G. Q. Chen et D. Wang [31, Th 3.1] pour un résultat d'existence et d'unicité dans $C^1(\mathbb{R} \times [0, +\infty))$ basé sur la méthode des caractéristiques. Le résultat de Chen-Wang prouve que l'équation d'Euler compressible ne crée pas de chocs, pour des données initiales croissantes et $C^1(\mathbb{R})$ (Voir aussi M. Grassin, D. Serre [68] pour un résultat similaire dans H^m). Dans notre cas, on sait que les solutions de (2.7)-(2.3) sont uniformément Lipschitz en espace et en temps. Même si cette question de régularité meilleure que Lipschitz n'est pas abordée dans ce résultat, nous imaginons que nous pouvons atteindre une régularité $C^1(\mathbb{R} \times [0, +\infty))$ de la solution pour des données initiales $C^1(\mathbb{R})$.

Nous signalons que, A. Briani *et al.* [22] et R. Monneau, P. E. Souganidis [109] ont prouvé un résultat d'homogénéisation pour le système (2.7) lorsque on rajoute un terme qui oscille fortement en temps dans la vitesse. Le modèle homogénéisé est une équation de diffusion non-linéaire. Ce qui montre un aspect "diffusif" dans le système (2.7).

2.2.2 Estimation d'erreur discrète-continue

Nous nous intéressons maintenant à l'approximation numérique de la solution du système (2.7)-(2.3). Étant donné une taille de discrétisation Δx, Δt, on définit la grille,

$$\Xi = \{i\Delta x,\ i \in \mathbb{Z}\}, \quad \Xi_T = \Xi \times \{0, ..., N_T \Delta t\},$$

où N_T est la partie entière de $T/\Delta t$ et pour $k = \{+, -\}$ on note par $v_i^{k,n} = v^k(x_i, t_n)$ la valeur de l'approximation numérique de la solution exacte de ρ^k au point (x_i, t_n) avec $x_i = i\Delta x$ et $t_n = n\Delta t$.

Nous allons maintenant introduire le schéma numérique. La difficulté principale vient du terme non-local qui nécessite la connaissance de la solution que l'on est en train de calculer. Pour résoudre ce problème, on fixe la solution $v_i^n = (v_i^{+,n}, v_i^{-,n})$ sur chaque intervalle de temps $[t_n, t_{n+1})$ et on applique le schéma monotone suivant, $\forall k \in \{+, -\}$

$$v_{0,i}^k = v_0^k(x_i) = \tilde{\rho}_0^k(x_i), \tag{2.9}$$

où $\tilde{\rho}_0^k$ est une approximation de ρ_0^k ;

$$v_i^{k,n+1} = v_i^{k,n} + \Delta t\ C_k^{\Delta}[v](x_i, t_n) \begin{cases} E^+\left(D^+ v_i^{k,n}, D^- v_i^{k,n}\right) & \text{si } C_k^{\Delta}[v](x_i, t_n) \geq 0, \\ E^-\left(D^+ v_i^{k,n}, D^- v_i^{k,n}\right) & \text{sinon,} \end{cases} \tag{2.10}$$

où

$$C_k^{\Delta}[v](x_i, t_n) = -k\left(v_i^{+,n} - v_{-,i}^{-,n} + L^{\Delta}[v](t_n)\right)$$

et le terme non-local $L^{\Delta}[v](t_n)$ est donné par

$$L^{\Delta}[v](t_n) = \sum_{i=0}^{N_x - 1} \Delta x \left(v^+(x_i, t_n) - v^-(x_i, t_n)\right),$$

où N_x est la partie entière de $1/\Delta x$. Ici, $E^{\pm}$ sont les approximations de la norme euclidienne proposées par S. Osher et J. A. Sethian [116], on peut également utiliser celles proposées par E. Rouy, A. Tourin [121] :

$$E^+(P,Q) = \left(\max(P,0)^2 + \min(Q,0)^2\right)^{\frac{1}{2}},$$

$$E^-(P,Q) = \left(\min(P,0)^2 + \max(Q,0)^2\right)^{\frac{1}{2}}.$$

Les termes $D^+v_i^{k,n}$, $D^-v_i^{k,n}$ sont des approximations appropriées du gradient de $v^{k,n}$ pris au point x_i :

$$D^+v_i^{k,n} = \frac{v_{i+1}^{k,n} - v_i^{k,n}}{\Delta x},$$

$$D^-v_i^{k,n} = \frac{v_i^{k,n} - v_{i-1}^{k,n}}{\Delta x}.$$

Finalement, on suppose que la condition CFL suivante est vérifiée

$$\Delta t \leq \frac{1}{2K_0}\Delta x \tag{2.11}$$

où

$$K_0 = 2\|\rho_0^+ - \rho_0^-\|_{L^\infty(\mathbb{R})} + 2.$$

Nous avons alors l'estimation d'erreur suivante :

Théorème 2.7 (Estimation d'erreur discrète-continue, *[48, Théorème 1.3]*) *Soit $T \geq 0$. On suppose que $\Delta x + \Delta t \leq 1$ et que la condition CFL (2.11) est vérifiée.*

Alors, il existe une constante $K > 0$ dépendante seulement de $\|\rho_0^+ - \rho_0^-\|_{L^\infty(\mathbb{R})}$ et $\max\limits_{k\in\{+,-\}} \left\|\frac{\partial \rho_0^k}{\partial x}\right\|_{L^\infty(\mathbb{R})}$ telle que l'estimation d'erreur entre la solution continue ρ^k du système (2.7)-(2.3) et son approximation numérique v^k, solution du schéma aux différences-finies (2.10) est donnée par :

$$\max_{k\in\{+,-\}} \sup_{\Xi_T} |\rho^k - v^k| \leq K\left((T+\sqrt{T})\,(\Delta x + \Delta t)^{1/2} + \max_{k\in\{+,-\}} \sup_{\Xi} |\rho_0^k - v_0^k|\right)$$

sous l'hypothèse complémentaire

$$K\left((T+\sqrt{T})(\Delta x + \Delta t)^{\frac{1}{2}} + \max_{k\in\{+,-\}} \sup_{\Xi}(\rho_0^k - v_0^k)\right) \leq 1.$$

Pour montrer ce théorème, on utilise la même méthode que pour le cas continu, c'est-à-dire qu'on considère la solution approximée du système (2.7)-(2.3) comme un point fixe d'un système local. Cette preuve est inspirée de la preuve de O. Alvarez *et al.* [4] pour montrer une estimation d'erreur de type Crandall-Lions [39] entre la solution continue et son approximation numérique en temps court pour des équations d'Hamilton-Jacobi monotones. Ici, nous avons montré le même genre d'estimation d'erreur dans le cadre d'un système, en utilisant le fait que nous avons un principe de comparaison dans le cas $\alpha = 0$. Puis par point fixe, nous déduisons le résultat pour le système non-local.

Nous nous référons également à E. R. Jakobsen, K. H. Karlsen [86] et E. R. Jakobsen, K. H. Karlsen, N. H. Risebro [87] qui ont prouvé une estimation d'erreur pour un système faiblement couplé de la forme

$$\frac{\partial u^i}{\partial t}(x,t) + H_i(t,x,u^i,Du^i) = G_i(t,x,u) \quad \text{sur } \mathbb{R}^N \times (0,T), \tag{2.12}$$

pour tout $i = 1, ..., M$. Leur estimation d'erreur est en $O(\Delta t)$ pour un algorithme de "splitting" semi-discret qui approche la solution de (2.12). Cependant, ici nous obtenons une estimation d'erreur en $O(\sqrt{\Delta t + \Delta x})$ parce que nous discrétisons aussi en espace.

2.2.3 Simulations numériques

Dans cette section, nous nous intéressons à l'évolution des densités de dislocations en dimension 1. Nous présentons quelques simulations numériques pour le sous-modèle (2.7)-(2.3), discrétisé par le schéma (2.10). Dans cette simulation, nous avons choisi un exemple de concentration des densités de dislocations, où les densités de dislocations sont initiallement périodiques, et égales à zéro dans des sous-intervalles de l'intervalle $[0,1[$ (voir Figure 4). Cette condition initiale signifie qu'il existe des régions sans dislocations et d'autres régions avec une concentration de dislocations.

Intuitivement, les dislocations se dispersent uniformément dans tout le cristal comme le montre la Figure 6. Finalement, les densités de dislocations deviennent constantes.

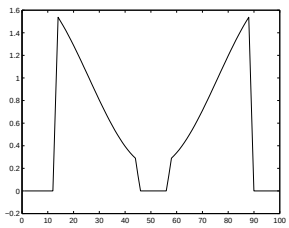

FIG. 4 – Densité de dislocations $\dfrac{\partial \rho_0^+}{\partial x} = \dfrac{\partial \rho_0^-}{\partial x}$ mais $\rho_0^+ \neq \rho_0^-$.

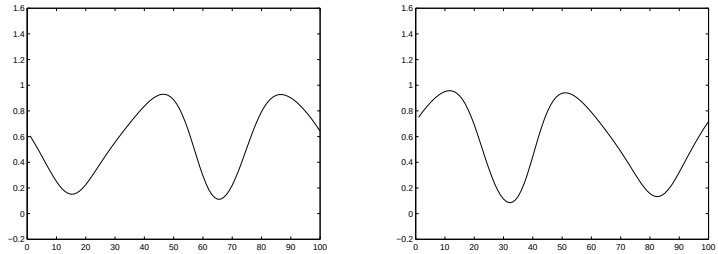

FIG. 5 – À gauche : densité $\dfrac{\partial \rho^+}{\partial x}(.,1)$; à droite : densité $\dfrac{\partial \rho^-}{\partial x}(.,1)$.

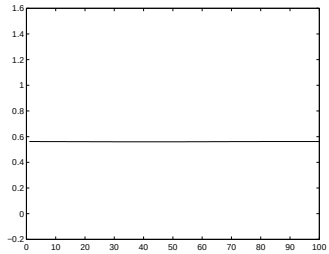

FIG. 6 – Densité de dislocations $\dfrac{\partial \rho^+}{\partial x}(.,3) = \dfrac{\partial \rho^-}{\partial x}(.,3)$.

2.3 Résultat d'existence globale et quelques résultats d'unicité pour une classe de systèmes hyperboliques diagonalisables

Dans ce qui précède, nous avons présenté des résultats concernant un système (2×2) unidimensionnel. Dans cette sous-section, nous allons exposer un résultat plus général pour des systèmes $(M \times M)$ hyperboliques diagonalisables, pour tout $M \in \mathbb{N}$. Plus précisément, nous présenterons un résultat d'existence globale et dans un cadre particulier quelques résultats d'unicité.

Remarque 2.8
Ce genre de systèmes hyperboliques diagonalisables apparaît naturellement dans la modélisation de la dynamique des densités de dislocations dans le cas de plusieurs directions de glissement. Nous renvoyons à la section 8 chapitre 4 pour une dérivation physique du modèle. Ceci était notre première motivation pour étudier ces systèmes.

Tout d'abord, nous nous intéressons à la solution $u(x,t) = (u^i(x,t))_{i=1,\dots,M}$, où M est un nombre entier, des systèmes hyperboliques qui sont diagonales, c'est-à-dire

$$\partial_t u^i + a^i(u)\partial_x u^i = 0 \quad \text{sur} \quad \mathbb{R} \times (0,T) \quad \text{et pour} \quad i = 1, ..., M, \tag{2.13}$$

avec des données initiales :

$$u^i(0,x) = u_0^i(x), \qquad x \in \mathbb{R}, \quad \text{pour } i = 1, \ldots, M. \tag{2.14}$$

Pour des nombres réels $\alpha^i \leq \beta^i$, nous considérons la boîte

$$U = \Pi_{i=1}^M [\alpha^i, \beta^i]. \tag{2.15}$$

Nous considérons une fonction donnée $a = (a^i)_{i=1,...,M} : U \to \mathbb{R}^M$, qui satisfait les régularités suivantes :

$$(C1) \quad \begin{cases} \text{la fonction } a \in C^\infty(U), \\ \\ \text{il existe} \quad M_0 > 0 \quad \text{telle que pour} \quad i = 1, ..., M, \\ |a^i(u)| \leq M_0 \quad \text{pour tout} \quad u \in U, \\ \\ \text{il existe} \quad M_1 > 0 \quad \text{telle que pour} \quad i = 1, ..., M, \\ |a^i(v) - a^i(u)| \leq M_1 |v - u| \quad \text{pour tout} \quad v, u \in U. \end{cases}$$

Avant d'énoncer le premier résultat de cette section, nous allons présenter l'idée principale. Cette idée est basée sur une nouvelle estimation sur l'entropie du gradient. Plus précisément, pour $w \geq 0$, nous considérons la fonction entropique suivante :

$$\bar{f}(w) = w \ln w.$$

Maintenant, pour toute fonction test positive $\varphi \in C_c^1(\mathbb{R} \times [0, +\infty))$, nous définissons l'entropie du gradient suivante, avec $w^i := \partial_x u^i$:

$$\bar{N}(t) = \int_{\mathbb{R}} \varphi \left(\sum_{i=1,...,M} \bar{f}(w^i) \right) dx. \tag{2.16}$$

Il est très naturel d'introduire une telle quantité $\bar{N}(t)$, car dans le cas $\varphi \equiv 1$ elle n'est rien d'autre que toute l'entropie du système de M particules des densités positives w^i. Alors il est formellement possible de déduire de (2.13), l'égalité suivante :

$$\frac{d\bar{N}}{dt}(t) + \int_{\mathbb{R}} \varphi \left(\sum_{i,j=1,...,M} a^i_{,j} w^i w^j \right) dx = R(t) \qquad \text{pour} \quad t \geq 0, \tag{2.17}$$

avec $a^i_{,j} = \dfrac{\partial}{\partial u^j} a^i$, et le reste

$$R(t) = \int_{\mathbb{R}} \left\{ (\partial_t \varphi) \left(\sum_{i=1,\dots,M} \bar{f}(w^i) \right) + (\partial_x \varphi) \left(\sum_{i=1,\dots,M} a^i \bar{f}(w^i) \right) \right\} dx,$$

où nous montrons seulement la dépendance en t dans les intégrales. Nous remarquons qu'en particulier ce reste est nul si $\varphi \equiv 1$.

Pour avoir un signe dans la partie gauche de l'égalité, nous avons supposé que, pour tout $u \in \mathbb{R}^M$, la matrice $(a^i_{,j}(u))_{i,j=1,\dots,M}$ est positive dans le cône positif, c'est-à-dire :

$$(C2) \left| \begin{array}{l} \text{pour tout} \quad u \in U, \quad \text{nous avons} \\ \\ \displaystyle\sum_{i,j=1,\dots,M} \xi_i \xi_j a^i_{,j}(u) \geq 0 \quad \text{pour tout} \quad \xi = (\xi_1, \dots, \xi_M) \in [0, +\infty)^M. \end{array} \right.$$

Nous remarquons que cette estimation sur l'entropie de gradient est le point clé dans notre résultat. Grâce à cette estimation, nous avons pu obtenir l'existence de solutions continues et croissantes pour le système (2.13).

Maintenant, nous considérons des données initiales croissantes en x et bornées dans cet espace d'entropie, que nous définirons plus tard. Plus précisément, chaque composant u_0^i des données initiales $u_0 = (u_0^1, \cdots, u_0^M)$ est supposé vérifier la propriété suivante :

$$(C3) \quad \left\{ \begin{array}{l} u_0^i \in L^\infty(\mathbb{R}), \\ \\ u_0^i \text{ est croissante}, \\ \\ \partial_x u_0^i \in L \log L(\mathbb{R}), \end{array} \right| \quad \text{pour } i = 1, \cdots, M,$$

où $L \log L(\mathbb{R})$ est l'espace de Zygmund suivant :

$$L \log L(\mathbb{R}) = \left\{ h \in L^1(\mathbb{R}) \text{ telle que } \int_{\mathbb{R}} |h| \ln(1 + |h|) < +\infty \right\}.$$

Cet espace est muni de la norme

$$\|h\|_{L \log L(\mathbb{R})} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}} \frac{|h|}{\lambda} \ln\left(1 + \frac{|h|}{\lambda}\right) \leq 1 \right\},$$

ce qui est dû à Luxembourg (voir R. A. Adams [2, (13), Page 234]). Cet espace $L\log L(\mathbb{R})$ découle naturellement de l'entropie (2.16).

Maintenant, nous annonçons le premier Théorème de cette sous-section. Nous signalons que dans le Théorème suivant, nous modifions la fonction entropique $\bar{f}$ par une autre fonction f, pour des raisons techniques qui apparaîtront dans la preuve.

Théorème 2.9 ***(Existence globale de solutions croissantes**, [49, Théorème 1.1])*
Sous les hypothèses $(C1)$, $(C2)$ et $(C3)$. Pour tout $T > 0$, nous avons :

i) **Existence de solution faible :**
Il existe une fonction u solution de (2.13)-(2.14) (au sens des distributions), où

$$u \in [L^\infty(\mathbb{R}\times(0,T))]^M \cap [C([0,T); L\log L(\mathbb{R}))]^M \text{ et } \partial_x u \in [L^\infty((0,T); L\log L(\mathbb{R}))]^M,$$

telle que pour presque tout $t \in [0,T)$ la fonction $u(\cdot,t)$ est croissante en x et vérifie l'estimation L^∞ suivante :

$$\|u^i(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \|u_0^i\|_{L^\infty(\mathbb{R})}, \qquad \text{pour } i = 1,\dots,M, \tag{2.18}$$

et l'estimation entropique de gradient :

$$\int_{\mathbb{R}} \sum_{i=1,\dots,M} f\left(\partial_x u^i(x,t)\right) dx + \int_0^t \int_{\mathbb{R}} \sum_{i,j=1,\dots,M} a^i_{,j}(u)\partial_x u^i(x,s)\partial_x u^j(x,s)\, dx\, ds \leq C_1, \tag{2.19}$$

où

$$f(x) = \begin{cases} x\ln(x) + \frac{1}{e} & \text{if } \ x \geq 1/e, \\ 0 & \text{if } \ 0 \leq x \leq 1/e, \end{cases} \tag{2.20}$$

et $C_1(T, M, M_1, \|u_0\|_{[L^\infty(\mathbb{R})]^M}, \|\partial_x u_0\|_{[L\log L(\mathbb{R})]^M})$.

ii) **Continuité de la solution :**
La solution u construite dans (i) est $C(\mathbb{R}\times[0,T))$. De plus, il existe un module de continuité $\omega(\delta,h)$, tel que pour tout $(x,t) \in \mathbb{R}\times(0,T)$ et pour tout $\delta, h \geq 0$, nous avons :

$$|u(x+h,t+\delta) - u(x,t)| \leq C_2\, \omega(\delta,h) \ \text{ avec } \ \omega(\delta,h) = \frac{1}{\ln(\frac{1}{h}+1)} + \frac{1}{\ln(\frac{1}{\delta}+1)}. \tag{2.21}$$

où $C_2(T, M_1, M_0, \|u_0\|_{[L^\infty(\mathbb{R})]^M}, \|\partial_x u_0\|_{[L\log L(\mathbb{R})]^M})$.

Notre preuve est basée sur cette remarque simple : si les données initiales satisfont $(C3)$ alors les solutions de (2.13)-(2.14) satisfont aussi $(C3)$ pour tout t. Ce qui semble nouveau ici, c'est l'inégalité d'entropie sur le gradient. La preuve du Théorème 2.9 est plutôt standard. Tout d'abord, nous régularisons les données initiales et le système par l'addition d'un terme de viscosité, nous montrons que ce système régularisé admet une solution classique. Nous prouvons l'estimation (2.18) et l'inégalité fondamentale d'entropie de gradient (2.19). Finalement, nous passons à la limite quand le terme de régularisation disparaît, nous obtenons l'existence d'une solution par un argument de compacité.

Remarque 2.10 (Systèmes hyperboliques diagonalisables)
Ce Théorème peut être également appliqué à des systèmes plus généraux que le système (2.13). Ces systèmes sont des systèmes hyperboliques diagonalisables. Nous renvoyons à la sous-section 1.3 chapitre 4 pour plus d'explications.

Maintenant, nous rappelons quelques résultats bien connus pour le système (2.13). Pour une équation de loi de conservation scalaire, qui correspond dans (2.13) au cas $M = 1$, $a^1(u) = h'(u)$ est la dérivée d'une certaine fonction de flux h, l'existence et l'unicité des solutions BV, ont été établies par O. A. Oleinik [113] en dimension 1 de l'espace. Le célèbre papier de S. N. Kruzhkov [93] couvre une classe plus générale de solutions L^∞, dans plusieurs dimensions de l'espace. Pour avoir une autre approche, basée sur la notion des solutions entropiques processus, voir R. Eymard et al. [52], ou pour une formulation cinétique voir P. L. Lions et al. [106].

Les résultats connus pour les systèmes 2×2 strictement hyperboliques, ce qui correspond dans (2.13) au cas $M = 2$ et

$$a^1(u^1, u^2) < a^2(u^1, u^2),$$

sont les résultats de P. D. Lax [98] et R. J. DiPerna [43,44] cités dans la sous-section 2.1.

Pour des systèmes généraux $M \times M$ strictement hyperboliques, ce qui correspond dans (2.13) au cas

$$a^1(u) < a^2(u) < \cdots < a^M(u), \tag{2.22}$$

S. Bianchini et A. Bressan ont prouvé dans [17] l'existence globale et l'unicité d'une solution BV, si les données initiales sont supposées à variation totale petite. Leur résultat est une généralisation de résultats de J. Glimm, prouvés initialement dans le cas des systèmes conservatifs, et généralisés ensuite par P. LeFloch et T. P. Liu [99, 100] pour le cas non-conservatif.

Ces résultats sont limités au cas des systèmes strictement hyperboliques. Ici, dans le Théorème 2.9, nous avons traité le cas des systèmes qui sont hyperboliques mais pas strictement hyperboliques. Dans la remarque suivante, nous montrons un exemple simple d'un système 2×2 où a^1 et a^2 se croisent.

Remarque 2.11 (Croisement des valeurs propres)
La condition (2.22) sur les a^i n'entre pas dans le cadre de notre travail (Théorème 2.9). Voici un exemple simple de système 2×2 hyperbolique mais pas strictement hyperbolique pour lequel notre Théorème s'applique. Nous considérons $u = (u^1, u^2)$ solution de

$$\left\{ \begin{array}{l} \partial_t u^1 + cos(u^2)\partial_x u^1 = 0, \\ \\ \partial_t u^2 + u^1 sin(u^2)\partial_x u^2 = 0, \end{array} \right| \quad sur\ \mathbb{R} \times (0,T). \tag{2.23}$$

Nous supposons que :

i) $u^1(-\infty) = 0$, $u^1(+\infty) = 1$ *et* $\partial_x u^1 \geq 0$,

ii) $u^2(-\infty) = -\frac{\pi}{2}$, $u^2(+\infty) = \frac{\pi}{2}$ *et* $\partial_x u^2 \geq 0$.

Ici $a^1(u^1, u^2) = cos(u^2)$ *et* $a^2(u^1, u^2) = u^1 sin(u^2)$ *se croisent au temps initial (et ensuite pour tout temps). Cependant, nous pouvons calculer*

$$(a^i_{,j}(u^1, u^2))_{i,j=1,2} = \begin{pmatrix} 0 & -sin(u^2) \\ sin(u^2) & u^1 cos(u^2) \end{pmatrix},$$

ce qui satisfait $(C2)$ *(sous les hypothèses (i) et (ii)). De plus, le Théorème 2.9 donne l'existence d'une solution pour (2.23) avec (i) et (ii).*

En se basant sur les mêmes types d'estimation sur l'entropie de gradient (2.19), nous allons montrer dans la section suivante un résultat d'existence et d'unicité pour le système bidimensionnel général de la dynamique des densités de dislocations.

Maintenant, nous présentons un résultat d'existence et d'unicité d'une solution $[W^{1,\infty}(\mathbb{R} \times [0,T))]^M$, dans un cas spécial pour simplifier la présentation. Plus précisément, nous supposons :

$$(C1') \quad a^i(u) = \sum_{j=1,\dots,M} A_{ij} u^j \text{ pour } i = 1, \dots, M \text{ et pour } u \in U,$$

$$(C2') \quad \sum_{i,j=1,\dots,M} A_{ij}\xi_i\xi_j \geq 0 \quad \text{pour tout} \quad \xi = (\xi_1, ..., \xi_M) \in [0, +\infty)^M.$$

Théorème 2.12 (Existence et unicité d'une solution $W^{1,\infty}$ pour $A = (A_{ij})_{i,j=i=1,\dots,M}$ particulière, *[49, Théorème 1.4])*
Soit $T > 0$. Sous l'hypothèse $(C1')$ et pour toutes données initiales $u_0 \in [W^{1,\infty}(\mathbb{R})]^M$ croissantes, le système (2.13)-(2.14) admet une unique solution $u \in [W^{1,\infty}(\mathbb{R} \times [0,T))]^M$, dans les cas suivants :

i) $M \geq 2$ et $A_{ij} \geq 0$, pour tout $j \geq i$.
ii) $M \geq 2$ et $A_{ij} \leq 0$, pour tout $i \neq j$ et $(C2')$. De plus, pour tout $(x,t) \in \mathbb{R} \times [0,T)$ nous avons :

$$\sum_{i=1,\dots,M} \partial_x u^i(x,t) \leq \sup_{y\in\mathbb{R}} \sum_{i=1,\dots,M} \partial_x u_0^i(y). \tag{2.24}$$

Remarque 2.13 (Le cas $M = 2$)
Particulièrement pour $M = 2$, si $(C1')$ et $(C2')$ vérifiées, nous avons par le Théorème 2.12 l'existence et l'unicité d'une solution dans $[W^{1,\infty}(\mathbb{R} \times [0,T))]^2$ pour (2.13)-(2.14).

Dans ces cas particuliers de la matrice A, nous pouvons montrer que $\partial_x u^i$ pour tout $i = 1, \dots, M$, sont bornés sur $\mathbb{R} \times [0,T)$. Grâce à cette meilleure estimation sur $\partial_x u^i$, nous déduisons une borne Lipschitz sur le champ de vitesse Au, qui nous permet d'obtenir l'unicité de la solution.

Dans un cas particulier où $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, nous avons montré dans les sous-sections précédentes 2.1 et 2.2, l'existence et l'unicité d'une solution de viscosité Lipschitz et un résultat similaire dans $H^1_{loc}(\mathbb{R} \times [0,T))$.

Nous signalons qu'ici dans le Théorème 2.9 et le Théorème 2.12, nous avons considéré un choix particulier de systèmes pour simplifier de la présentation. Ces deux Théorèmes pourraient être également généralisés pour d'autres systèmes, voir sous-section 1.5 chapitre 4 pour plus de détails.

3 Seconde partie : modèle bidimensionnel

Dans cette section, nous allons mettre en œuvre cette estimation d'entropie sur le gradient, présentée dans la sous-section 2.3. En se servant de celle-ci, nous allons exposer un résultat d'existence pour le modèle bidimensionnel (1.1) de la dynamique des densités de dislocations, en explicitant le noyau c_0.

3.1 Résultat d'existence globale dans un espace d'entropie

Comme nous l'avons annoncé précèdemment, le terme de convolution $c_0 * (\rho^+ - \rho^-)$ représente la force créée par le champ élastique généré par les lignes de dislocations. D'un point de vue pratique, après avoir résolu l'équation d'élasticité dans un matériau isotrope (c'est-à-dire ses propriétés sont les mêmes dans toutes les directions), nous pouvons montrer que le champ de vitesse dans les équations $c_0 * (\rho^+ - \rho^-)$ peut être exprimé comme certaines transformations de Riesz de $\rho^+ - \rho^-$ (on renvoit à la section 2 du chapitre 5 pour une dérivation détaillée de ce modèle). Ce qui nous ramène finalement à étudier le système des équations de transport non-linéaire et non-local suivant :

$$\begin{cases} \dfrac{\partial \rho^+}{\partial t}(x_1, x_2, t) = - \quad (R_1^2 R_2^2 \left(\rho^+(\cdot, t) - \rho^-(\cdot, t)\right)(x_1, x_2)) \dfrac{\partial \rho^+}{\partial x_1}(x_1, x_2, t) \quad \text{sur } \mathbb{R}^2 \times (0, T) \\ \dfrac{\partial \rho^-}{\partial t}(x_1, x_2, t) = \quad\quad (R_1^2 R_2^2 \left(\rho^+(\cdot, t) - \rho^-(\cdot, t)\right)(x_1, x_2)) \dfrac{\partial \rho^-}{\partial x_1}(x_1, x_2, t) \quad \text{sur } \mathbb{R}^2 \times (0, T) \end{cases} \tag{3.25}$$

Nous rappelons que les inconnues de ce système sont les scalaires ρ^+ et ρ^+ et que les densités de dislocations de type $\pm$ sont présentées, ici, par $\dfrac{\partial \rho^\pm}{\partial x_1}$. Les opérateurs R_1 (resp. R_2) sont les transformations de Riesz associées à x_1 (resp. x_2) définies comme :

Définition 3.1 (Transformations de Riesz)
Soient $p > 1$ et $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, le carré $[0,1) \times [0,1)$ périodique. Si $f \in L^p(\mathbb{T}^2)$, nous définissons R_i pour $i \in \{1,2\}$ comme les transformations de Riesz sur $\mathbb{T}^2$ telles que leurs coefficients en série de Fourier sont données par :

i) $c_{(0,0)}(R_i f) = 0$.

ii) $c_k(R_i f) = \dfrac{k_i}{|k|} c_k(f)$ *pour* $k = (k_1, k_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$,

où $c_k(f) = \displaystyle\int_{\mathbb{T}^2} f(x_1, x_2) e^{-2\pi i (k_1 x_1 + k_2 x_2)} dx_1 dx_2$.

Le système (3.25) est complété par les conditions initiales suivantes :

$$\rho^\pm(x_1, x_2, t = 0) = \rho_0^\pm(x_1, x_2) = \rho_0^{\pm, per}(x_1, x_2) + L_0 x_1, \tag{3.26}$$

où $\rho_0^{\pm,per}$ sont des fonctions 1-périodiques en x_1 et en x_2. Comme dans le cas unidimensionnel, nous avons modèlisé une distribution périodique de dislocations + et − avec une densité totale L_0 de chaque espèce par période spatiale de longueur 1 en x_1 et en x_2. La périodicité revient à considérer un domaine infini et à éviter les problêmes dus aux conditions aux bords.

Nous remarquons que la difficulté de ce problême est la présence d'un terme non-local dans le champ de vitesse. Ce dernier ne nous permet pas d'obtenir un principe de maximum contrairement au cas unidimensionnel. Ceci entraînera des difficultés pour définir le produit dans le terme bilinéaire

$$R_1^2 R_2^2 \left(\rho^+ - \rho^-\right) \frac{\partial \rho^\pm}{\partial x_1},$$

du système (3.25), même avec une estimation sur l'entropie de gradient de type (2.19). En particulier, nous aurons également des difficultés pour avoir assez de compacité afin d'assurer le passage à la limite.

Avant d'énoncer le résultat, nous définissons l'espace d'entropie $L \log L$ sur $\mathbb{T}^2$ et nous présentons une proposition qui nous permet de bien définir le produit dans le terme bilinéaire du système et ensuite dans l'énoncé du Théorème principal de cette sous-section.

Définition 3.2 (L'espace $L \log L$)
Soit $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. On définit $L \log L(\mathbb{T}^2)$ comme l'espace de Zygmund (voir C. Bennett et R. Sharpley [16, Page 243]) :

$$L \log L(\mathbb{T}^2) = \left\{ h \in L^1(\mathbb{T}^2) \text{ telle que } \int_{\mathbb{T}^2} |h| \ln\left(e + |h|\right) < +\infty \right\}.$$

Cet espace est muni de la norme de Luxembourg suivante

$$\|h\|_{L \log L(\mathbb{T}^2)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{T}^2} \frac{|h|}{\lambda} \ln\left(e + \frac{|h|}{\lambda}\right) \leq 1 \right\},$$

Proposition 3.3 (Signification du terme bilinéaire)
Soient $T > 0$, f et g deux fonctions définies sur $\mathbb{T}^2 \times (0,T)$, telles que $f \in L^1((0,T); H^1(\mathbb{T}^2))$ et $g \in L^\infty((0,T); L \log L(\mathbb{T}^2))$ alors,

$$fg \in L^1(\mathbb{T}^2 \times (0,T)).$$

La démonstration de cette proposition est donnée dans la sous-section 4.2 chapitre 5.

Théorème 3.4 (Existence globale pour le système (3.25)-(3.26), *[25, Théorème 1.4]*)
Pour tous $T, L_0 > 0$, et pour toutes données initiales $\rho_0^\pm \in L^2_{loc}(\mathbb{R}^2)$ vérifiant les conditions suivantes :

(A1) $\rho_0^\pm(x_1+1, x_2) = \rho_0^\pm(x_1, x_2) + L_0$, *presque partout dans $\mathbb{R}^2$, (fonction linéaire + fonction périodique en x_1),*

(A2) $\rho_0^\pm(x_1, x_2+1) = \rho_0^\pm(x_1, x_2)$, *presque partout dans $\mathbb{R}^2$, (périodique en x_2),*

(A3) $\dfrac{\partial \rho_0{}^\pm}{\partial x_1} \geq 0$, *presque partout dans $\mathbb{R}^2$, ($\rho_0^\pm$ croissante en x_1),*

(A4) $\left\| \dfrac{\partial \rho_0{}^\pm}{\partial x_1} \right\|_{L \log L(\mathbb{T}^2)} \leq C$, *avec $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$,*

le système (3.25)-(3.26) admet des solutions $\rho^\pm \in L^\infty\left((0,T); L^2_{loc}(\mathbb{R}^2)\right) \cap C\left([0,T); L^1_{loc}(\mathbb{R}^2)\right)$ au sens des distributions. Ces solutions vérifient $(A1)$, $(A2)$, $(A3)$ et $(A4)$ pour presque tout $t \in (0,T)$. De plus, nous avons :

(P1) $R_1^2 R_2^2 \left(\rho^+ - \rho^-\right) \in L^2\left((0,T); H^1_{loc}(\mathbb{R}^2)\right)$.

Remarque 3.5 (Terme bilinéaire)
Il est clair qu'ici le terme bilinéaire est toujours défini via $(P1)$ et la Proposition 3.3.

Le point clé de la démonstration est l'estimation d'entropie suivante sur le gradient dans l'espace $L \log L(\mathbb{T}^2)$:

$$\begin{aligned} \int_{\mathbb{T}^2} \sum_{\pm} \frac{\partial \rho^\pm}{\partial x_1}(\cdot, t) \ln\left(\frac{\partial \rho^\pm}{\partial x_1}(\cdot, t)\right) \quad &+ \int_0^t \int_{\mathbb{T}^2} \left(R_1 R_2 \left(\frac{\partial \rho^+}{\partial x_1} - \frac{\partial \rho^-}{\partial x_1}\right)\right)^2 \\ &\leq \int_{\mathbb{T}^2} \sum_{\pm} \frac{\partial \rho_0^\pm}{\partial x_1} \ln\left(\frac{\partial \rho_0^\pm}{\partial x_1}\right), \end{aligned} \tag{3.27}$$

Grâce à cette estimation, on peut avoir un double contrôle, le premier sur le terme $R_1^2 R_2^2 \left(\rho^+ - \rho^-\right)$ dans $L^2\left((0,T); H^1(\mathbb{T}^2)\right)$ en utilisant quelques propriétés des transformations de Riesz et le deuxième sur le terme $\dfrac{\partial \rho^\pm}{\partial x_1}$ dans $L^\infty((0,T); L \log L(\mathbb{T}^2))$. Ce double contrôle nous permet d'une part, de bien définir le produit

$$R_1^2 R_2^2 \left(\rho^+ - \rho^-\right) \frac{\partial \rho^\pm}{\partial x_1},$$

à l'aide de la Proposition 3.3 et d'autre part d'avoir assez de compacité dans ce terme bilinéaire, ce qui va nous conduire à l'existence globale d'une solution par compacité.

Ici, nous ne pouvons pas appliquer les théories de R. J. DiPerna et P. L. Lions [45] et L. Ambrosio [8] concernant l'existence et l'unicité d'une solution renormalisée pour l'équation de transport, pour une simple raison que notre champ de vitesse vérifie

$$R_1^2 R_2^2 \left(\rho^+ - \rho^-\right) \in L^2((0,T); H^1_{loc}(\mathbb{R}^2)),$$

et nous n'avons aucune autre estimation sur la divergence du champ de vecteurs. Cela signifie que la divergence n'est pas bornée dans notre étude. Nous avons contourné, ici, la difficulté en montrant des estimations sur le gradient de solutions. Ce genre d'estimations n'était pas demandé dans la théorie de DiPerna-Lions.

Évoquons maintenant d'autres modèles posant de problèmes de régularités dans le champ de vecteurs comme dans notre étude. Par contre, dans ces modèles nous pouvons appliquer la théorie de solutions renormalisées puisqu'ils possédent plus de régularités dans le champ de vecteur que dans le système (3.25). Voici quelques exemples : le modèle de Vlasov-Poisson (voir par exemple J. Nieto *et al.* [111]) et le modèle de type supraconductivité étudié par N. Masmoudi *et al.* [107] et par L. Ambrosio *et al.* [9]. Ces derniers modèles ont été dérivés de certains modèles de Vlasov-Poison-Fokker-Planck (voir par exemple T. Goudon *et al.* [67], et P. Chavanis *et al.* [30] pour avoir une vue d'ensemble des modèles qui leurs sont semblables). Mentionnons aussi que le modèle (3.25) est lié à l'équation de Vlasov-Navier-Stokes, voir T. Goudon *et al.* [65, 66].

Nous pouvons aussi remarquer que dans le cas où nous multiplions le second terme des deux équations du système (3.25) par (-1), nous obtenons un système de type quasi-géostrophique qui est de la forme :

$$\begin{cases} \dfrac{\partial \theta}{\partial t} + u \cdot \nabla \theta &= 0, \qquad \text{sur } \mathbb{R}^2 \times (0,T), \\ \\ u = (u_1, u_2) &= (-R_2\theta, R_1\theta). \end{cases}$$

où θ est un scalaire. En ce qui concerne ce sytème, nous nous référons à P. Constantin *et al.* [34,35] pour certains résultats numériques en $2D$. Nous citons également J. Wu [133, Th 4.1] pour un résultat d'existence et d'unicité local bidimensionnel dans l'espace d'Hölder et à A. Córdoba, D. Córdoba [36], D. Chae, A. Córdoba [29] pour des résultats de singularité en temps fini en dimension 1. L'équation en dimension 1 revient à remplacer la transformation de Riesz par une transformation de Hilbert. Dans notre cas, si nous multiplions le système (3.25) par -1, on obtiendra une inversion de signe dans l'estimation de l'entropie du gradient (3.27).

Brève littérature autour de la dynamique des lignes de dislocations :

Dans ce mémoire, nous avons étudié la dynamique des densités de dislocations. Il existe également plusieurs travaux concernant l'étude de la dynamique des lignes de dislocations. Signalons au passage quelques résultats récents. La dynamique d'une ligne de dislocation a été modélisée par O. Alvarez *et al.* dans [7] par une équation d'Hamilton-Jacobi non-locale, où les auteurs ont pu montrer l'existence et l'unicité locale d'une solution de viscosité. Concernant cette dynamique, nous renvoyons également à N. Forcadel [55] pour un résultat similaire avec un terme de courbure moyenne. Sous certaines hypothèses de monotonie sur la vitesse, O. Alvarez *et al.* [3], G. Barles, O. Ley [15] et P. Cardaliaguet, C. Marchi [26] ont montré, avec des méthodes différentes l'existence et l'unicité globale de solution de viscosité. Nous renvoyons aussi à G. Barles *et al.* [14] et N. Forcadel et A. Monteillet [58] pour un résultat d'existence globale d'une solution faible sous des hypothèses très générales. Toujours dans le cadre de la dynamique des lignes de dislocations, on peut trouver dans O. Alvarez *et al.* [4,5], A. Ghorbel, R. Monneau [62] et N. Forcadel [56] quelques résultats numériques, ainsi que des résultats d'homogénéisation dans C. Imbert *et al.* [79, 80], A. Ghorbel et al. [61] et N. Forcadel *et al.* [57].

Guide de lecteur de la thèse :

Dans le chapitre 1, nous présentons la modélisation physique du modèle bidimensionnel de Groma-Balogh introduit précédemment. Dans le chapitre 2, nous montrons un résultat d'existence et d'unicité globale dans $H^1(\mathbb{R} \times [0, T))$ pour un sous-modèle unidimensionnel simplifié. Ensuite, dans le chapitre 3, nous prouvons l'existence et l'unicité globale d'une solution de viscosité et une estimation d'erreur discrète-continue pour ce sous-modèle. Dans le chapitre 4, nous traitons, dans un cadre plus général, des systèmes unidimensionnels où nous prouvons un résultat d'existence et quelques résultats d'unicité. Dans le chapitre 5, nous montrons l'existence globale de solutions pour le modèle de Groma-Balogh bidimensionnel. Nous présentons dans le chapitre 6 quelques simulations numériques pour ce modèle bidimensionnel. Nous finirons par une conclusion et nous énoncerons quelques perspectives de travail futur.

Chapitre 1

Modélisation des la dynamique de densités de dislocations

Nous commençons ce chapitre en donnant un bref historique des travaux concernant les dislocations, puis nous énoncerons les propriétés classiques des dislocations et enfin nous présenterons la modèlisation du modèle de Groma-Balogh [71].

1 Historique

Comme nous l'avons déjà indiqué, les dislocations sont des lignes de défauts dans les cristaux. La théorie des dislocations a été, à l'origine, développée par V. Volterra en 1907 [131]. D'un point de vue mécanique, dans les années trente, ces défauts ont été introduits par G. I. Taylor [128], E. Orowan [115] et M. Polanyi [119], comme l'explication principale à l'échelle microscopique des déformations plastiques à l'échelle macroscopique des cristaux. En 1956, les premières observations directes des dislocations en microscopie électronique sont faites par W. Bollman [20] et P. B. Hirsch, L. Horne, M. S. Whelan [73]. Ces observations ont permis de tester un certain nombre de prédictions théroriques. On trouve une description de la statique des dislocations dans les traités classiques, achevés dès la fin des années soixante, voir F. R. N. Nabarro [110] et J. R. Hirth, L. Lothe [74]. On pourra aussi consulter R. W. Lardner [97] pour une présentation plus mathématique.
Au milieu des années 80, on observe un développement remarquable des méthodes de simulations de la dynamique des lignes de dislocations (voir V. Bulatov *et al.* [41]), en particulier motivé par l'augmentation de la puissance de calcul des ordinateurs. Ces simulations ont permis aux mécaniciens de mieux comprendre le comportement mécanique issu de la déformation plastique des matériaux cristallins.

La quantité de dislocations dans un cristal est caractérisée par sa densité, définie par le nombre des lignes de dislocations qui traversent une section unitaire. Dans

les années cinquante, J. F. Nye [112], E. Köner [94], K. Kondo [89] et B. A. Bilby *et al.* [19] ont été les premiers à établir une étude théorique de la densité de dislocations. Ces études ont offert une meilleure compréhension mécanique des matériaux à une échelle "mésoscopique", plus grande que l'échelle de dislocations. Ces travaux ont été en particulier pertinents dans le cas où on a une forte concentration de dislocations. Ensuite, plusieurs tentatives ont été effectuées pour décrire une évolution continue des densités de dislocations. Nous renvoyons à D. Kuhlmann-Wilsdorf *et al.* [95], D. Holt [77], D. Walgraef, E. Aifantis [132] et J. Kratochvil *et al.* [1,92]. Leurs théories ont été basées sur des hypothèses pouvant être remises en question, et sur des pararmètres difficiles à déterminer.

Dans une géométrie particulière, I. Groma *et al.* [69,71,72] ont décrit une évolution bidimensionnelle continue des densités de dislocations (la dynamique) par un modèle qui couple fortement les équations de l'élasticité linéaire avec un système d'équations de type transport non-local. Notre étude a été focalisée sur ce modèle.

2 Propriétés des dislocations

Généralement, il existe deux types de dislocations : les *"dislocations coins"* et les *"dislocations vis"*.

Regardons d'abord un mécanisme fictif pour l'introduction d'une dislocation dans un cristal. On coupe un cristal parfait selon un demi-plan $ABCD$ (voir Figure 1.1). On translate à droite la partie du haut par un vecteur $\vec{b}$ reliant deux atomes voisins (voir Figure 1.1). On rétablit les liaisons atomiques autour de la ligne AD, néanmoins, la translation induit un défaut dans l'arrangement des atomes autour de la droite BC. En particulier, les atomes le long de BC, n'ont pas retrouvé leurs liaisons, il s'agit d'un défaut linéaire appelé une "dislocation coin". On peut également créer la dislocation coin en enfonçant un demi-plan atomique vertical comme un coin dans le cristal parfait. La dislocation coin est alors la frontière du demi-plan supplémentaire, là où la structure cristalline est fortement déformée. Ce type de dislocation a été imaginé simultanément et indépendamment par G. I. Taylor [128], E. Orowan [115] et M. Polanyi [119] en 1934.

On peut aussi imaginer que la translation $\vec{b}$ soit parallèle au bord de la coupure BC (voir Figure 1.2). Ce défaut est appelé "dislocation vis". Une manière d'imaginer une dislocation vis est de faire une coupure sur un demi-plan d'un cristal parfait et de translater le quart d'espace supérieur dans une direction parallèle au bord du demi-plan de coupure et le demi-espace inférieur dans la direction opposée. La dislocation vis est alors représentée par le bord du demi-plan de coupure. Cette dislocation a

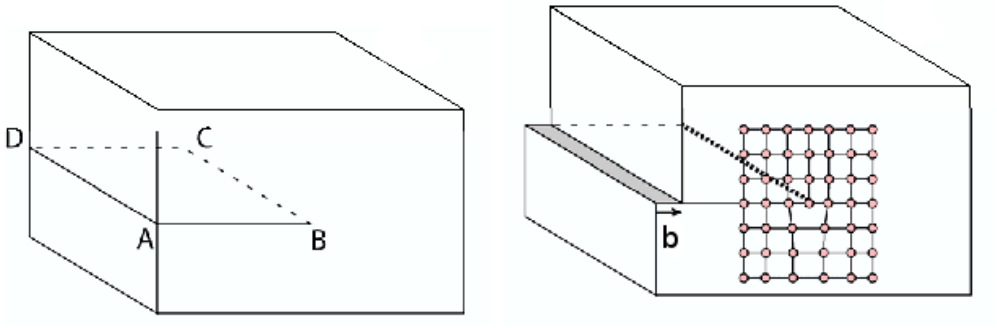

FIG. 1.1 – Dislocation coin

été imaginée par Burgers [23] en 1939.

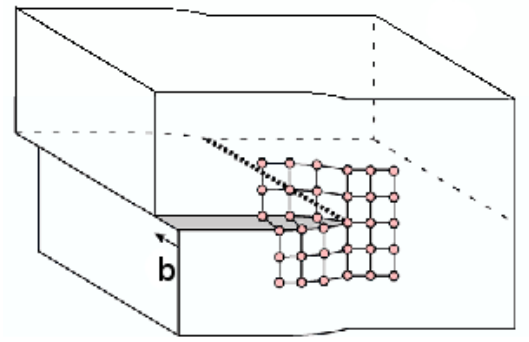

FIG. 1.2 – Dislocation vis

Une dislocation est alors caratérisée par ce vecteur $\vec{b}$, appelé vecteur de "Burgers", qui indique la direction du déplacement de la dislocation et dont la norme représente l'amplitude de la déformation qu'elle engendre.

Comme nous l'avons vu, la ligne de dislocation est perpendiculaire à $\vec{b}$ pour la dislocation coin et parallèle à $\vec{b}$ pour la dislocation vis. Nous remarquons que dans le cas général, la ligne de dislocation forme un angle arbitraire avec le vecteur de Burgers et nous avons donc une dislocation mixte.

3 Modélisation de la dynamique des densités de dislocations

Nous allons maintenant décrire la dynamique des densités de dislocations. Cette partie est inspirée de I. Groma, P. Balogh [71], mais la présentation de la modélisa-

tion est faite de manière un peu différente.

Tout d'abord, on se place dans un cas particulier où les lignes de dislocations sont des droites parallèles dans l'espace tridimensionnel se déplaçant dans la même direction. On suppose également que chaque dislocation se déplace suivant un vecteur de Burgers $\vec{b}$ perpendiculaire à la ligne de dislocation (c'est-à-dire "dislocation coin"). Dans cette géométrie particulière, les lignes de dislocations peuvent être vues comme des points dans le plan transversal $(\vec{b}, \vec{n})$ où $\vec{n}$ est le vecteur normal au plan de glissement (voir Figure 1.3).

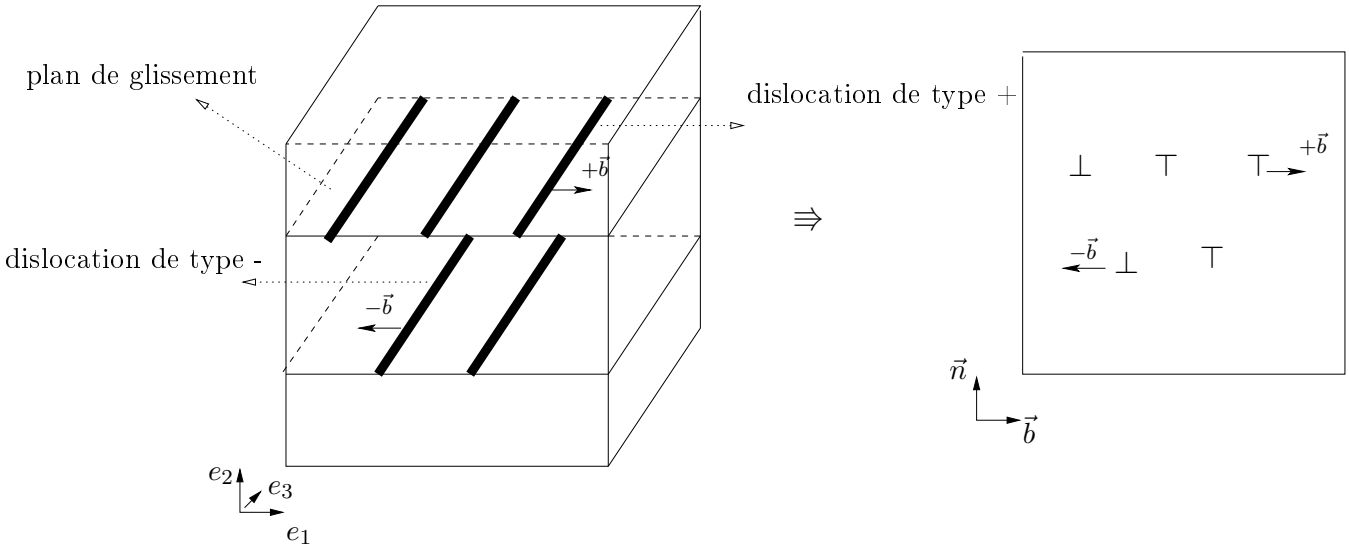

FIG. 1.3 – Passage du cas tridimensionnel au cas bidimensionnel

Étant donné que les dislocations glissent dans la même direction que le vecteur de Burgers $\vec{b}$, on remarque que pour un champ de vitesse donné, ces points peuvent se propager uniquement suivant deux vecteurs de Burgers $\pm\vec{b}$. De ce fait, nous avons deux types de dislocations, celles de type $(+)$ qui se propagent suivant le vecteur $+\vec{b}$ et celle de type $(-)$ qui se propagent suivant $-\vec{b}$.

Maintenant, on suppose pour simplifier que nous avons N dislocations de type $+$ et également N dislocations de type $-$, et on note par $\vec{r_i}^{\pm}$ la position d'une dislocation de type $\pm$ dans le plan $(\vec{b}, \vec{n})$. On introduit pour tout vecteur $\vec{r}$ du plan, la fonction de densité discrète de type $\pm$, définie par les distributions discrètes suivantes :

$$\theta^{\pm,d}(\vec{r}) = \sum_{i=1}^{N} \delta(\vec{r} - \vec{r_i}^{\pm}),$$

On remarque que pour $i = 1, \ldots, N$, nous avons :

$$\frac{d}{dt}\delta(\vec{r} - \vec{r_i}^{\pm}) = -\nabla \cdot \left[\frac{d}{dt}(\vec{r_i}^{\pm})\ \delta(\vec{r} - \vec{r_i}^{\pm})\right], \tag{3.1}$$

De plus, $\frac{d}{dt}\vec{r_i}^{\pm} = \pm(\vec{b}\ v)$ où v est la vitesse de la dislocation. Cette vitesse est proportionnelle à la force de Peach-Koehler résolue s'exerçant sur la dislocation. Dans le cas où il n'y a pas de contraintes extérieures, cette force est simplement la force créée par le champ élastique généré par la dislocation elle-même (voir M. Peach et J. S. Koehler [118]).

En additionnant (3.1) pour $i = 1, \ldots, N$, on conclut que

$$\frac{d}{dt}\theta^{\pm,d} = \mp\nabla \cdot \left[\vec{b}\ v\ \theta^{\pm,d}\right], \tag{3.2}$$

Les densités de dislocations continues $\theta^{\pm}$ peuvent être considérées comme la régularisation des densités discrètes $\theta^{\pm,d}$. Ici, nous supposons qu'elles vérifient également la même équation d'évolution de $\theta^{\pm,d}$. Alors, nous avons :

$$\frac{d}{dt}\theta^{\pm} = \mp\nabla \cdot \left[\vec{b}\ v\theta^{\pm}\right], \tag{3.3}$$

Nous allons maintenant calculer la force de Peach-Koeller v. Tout d'abord, on considère un cristal représenté par l'espace tout entier, soumis à l'élasticité linéaire et dont les coefficients d'élasticité sont donnés par $\Lambda = (\Lambda_{ijkl})_{i,j,k=1,2,3}$. On suppose que ces coefficients satisfassent la propriété de symétrie suivante :

$$\Lambda_{ijkl} = \Lambda_{jikl} = \Lambda_{ijlk} = \Lambda_{klij},$$

et l'hypotèse de coercitivité suivante pour $m > 0$

$$\sum_{i,j,k=1,2,3} \Lambda_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq m \sum_{i,j,k=1,2,3} \varepsilon_{ij}^2,$$

pour toutes matrices constantes $\varepsilon = (\varepsilon_{ij})_{i,j=1,2,3}$ symétriques, c'est-à-dire vérifiant $\varepsilon_{ij} = \varepsilon_{ji}$.

On note par $u = (u_1, u_2, u_3) : \mathbb{R}^3 \to \mathbb{R}^3$ le déplacement du cristal qu'on choisit "à moyenne nulle". De plus on note par $x = (x_1, x_2, x_3)$ les coordonnés d'un point dans une base orthogonale $(\vec{b}, \vec{n}, \vec{e_3})$. Nous définissons maintenant la déformation totale du cristal par :

$$\varepsilon(u) = \frac{1}{2}(\nabla u + {}^t\nabla u), \quad \text{c-à-d} \quad \varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2, 3.$$

Cette déformation totale est décomposée de deux parties :

$$\varepsilon(u) = \varepsilon^e(u) + \varepsilon^p,$$

telle que $\varepsilon^e(u)$ est la déformation élastique et ε^p est la déformation plastique du matériau, donnée par la formule suivante :

$$\varepsilon^p = \varepsilon^0 \gamma, \tag{3.4}$$

où ε^0 est la matrice de cisaillement définie dans le cas d'une seule direction de glissement où les dislocations se déplacent dans le plan $\{x_3 = 0\}$ par

$$\varepsilon^0 = \frac{1}{2}\left(\vec{b} \otimes \vec{n} + \vec{n} \otimes \vec{b}\right)$$

Ici, γ est la contrainte plastique résolue, elle sera précisée plus tard.

Les contraintes d'élasticité σ_{ij} pour $i, j = 1, 2, 3$ sont reliées à la déformation élastique par le loi de Hooke :

$$\sigma = \Lambda : \varepsilon^e(u), \quad \text{c-à-d} \quad \sigma_{ij} = \sum_{k,l=1,2,3} \Lambda_{ijkl} \varepsilon^e_{kl}(u), \tag{3.5}$$

et satisfont l'équation d'Euler-Lagrange

$$\operatorname{div} \sigma = 0. \tag{3.6}$$

Nous considérons une solution particulière où $u_3 = 0$. Étant donné que nous sommes dans une géométrie particulière où les dislocations sont des droites parallèles à la direction e_3, et en ayant considéré un problème invariant par translation dans la direction de x_3, alors notre étude est réduite à un problème bidimensionnel avec u_1, u_2 dépendant seulement de (x_1, x_2). Ainsi nous pouvons exprimer la contrainte plastique résolue γ comme :

$$\gamma = \rho^+ - \rho^-, \tag{3.7}$$

où les quantités ρ^+ et ρ^- sont des scalaires représentant respectivement les discontinuités de déplacement dûes aux dislocations glissant dans les directions $+\vec{b}$ et $-\vec{b}$. Les quantités $\vec{b} \cdot \nabla\rho^+$ et $\vec{b} \cdot \nabla\rho^-$ sont positives et représentent respectivement les densités de dislocations de vecteurs de Burgers $+\vec{b}$ et $-\vec{b}$.

Nous prenons en considération l'énergie élastique définie par

$$E^{el} = \frac{1}{2} \int_{\mathbb{R}^2} (\Lambda : \varepsilon^e(u)) : \varepsilon^e(u).$$

Nous calculons formellement la variation de l'énégie élastique associée aux dislocations présentées ici par $\rho^{\pm}$. Nous obtenons que la force de Peach -Koeller est donnée par (voir O. Alvarez *et al.* [7, Section 2.6] pour plus de détails) :

$$v = \nabla_{\rho^{\pm}} E^{el} = -(\sigma : \varepsilon^0). \tag{3.8}$$

En injectant (3.8) dans (3.3) et en prenant en compte les équations (3.6)-(3.5) et (3.4)-(3.7), on déduit que la dynamique des densités de dislocations est décrite par le système bidimensionnel suivant (voir I. Groma, P. Balogh [71]) :

$$\left\{\begin{array}{lll} div\ \sigma & = 0 & \text{sur } \mathbb{R}^2 \times (0,T), \\ \sigma & = \Lambda : (\varepsilon(u) - \varepsilon^p) & \text{sur } \mathbb{R}^2 \times (0,T), \\ \varepsilon(u) & = \frac{1}{2}\left(\nabla u + {}^t\nabla u\right) & \text{sur } \mathbb{R}^2 \times (0,T) \\ \varepsilon^p & = \varepsilon^0\left(\rho^+ - \rho^-\right) & \text{sur } \mathbb{R}^2 \times (0,T), \end{array}\right\} \text{(Élasticité)}$$
$$\left\{\begin{array}{ll} \theta^{\pm} & = \vec{b}.\nabla\rho^{\pm} \quad \text{sur } \mathbb{R}^2 \times (0,T), \\ \dfrac{\partial\theta^{\pm}}{\partial t} & = \pm\vec{b}.\nabla\left[(\sigma : \varepsilon^0)\theta^{\pm}\right] \quad \text{sur } \mathbb{R}^2 \times (0,T). \end{array}\right\} \text{(Transport)} \tag{3.9}$$

Dans notre étude nous avons traité le cas où nous intègrons la dernière équation de ce système (équation de transport) dans la direction du champ $\vec{b}.\nabla$, ce qui nous ramène à l'étude du système suivant :

$$\left\{\begin{array}{lll} div\ \sigma & = 0 & \text{sur } \mathbb{R}^2 \times (0,T), \\ \sigma & = \Lambda : (\varepsilon(u) - \varepsilon^p) & \text{sur } \mathbb{R}^2 \times (0,T), \\ \varepsilon(u) & = \frac{1}{2}\left(\nabla u + {}^t\nabla u\right) & \text{sur } \mathbb{R}^2 \times (0,T) \\ \varepsilon^p & = \varepsilon^0\left(\rho^+ - \rho^-\right) & \text{sur } \mathbb{R}^2 \times (0,T), \end{array}\right\} \text{(Élasticité)}$$
$$\left\{\dfrac{\partial\rho^{\pm}}{\partial t} = \pm(\sigma : \varepsilon^0)\vec{b}.\nabla\rho^{\pm} + g \quad \text{sur } \mathbb{R}^2 \times (0,T). \right\} \text{(Transport)} \tag{3.10}$$

où les inconnues sont $\rho^{\pm}$ et le déplacement $u = (u_1, u_2)$. La fonction g donnée, vérifiant $\vec{b}.\nabla g = 0$ est supposée nulle pour simplifier.

En utilisant la transformation de Fourier et un calcul explicite de la fonction de Green, puisque le déplacement u est choisi "à moyenne nulle", nous pouvons résoudre l'équation d'élasticité linéaire et nous pouvons calculer u en fonction de $\rho^+ - \rho^-$. Ceci nous permet d'éliminer l'équation d'élasticité en réécrivant le champ de vitesse $\sigma : \varepsilon^0$ de la manière suivante :

$$\sigma : \varepsilon^0 = -c_0 * (\rho^+ - \rho^-),$$

Nous renvoyons à O. Alvarez *et al.* [7, Section 2.6] pour plus de détails sur le calcul de c_0 et sur ses propriétés. Finalement, nous pouvons déduire

$$\frac{\partial \rho^{\pm}}{\partial t} = \mp \left(c_0 * (\rho^+ - \rho^-)\right) \vec{b} \cdot \nabla \rho^{\pm} \quad \text{sur } \mathbb{R}^2 \times (0, T).$$

Ce système est équivalent à celui énoncé dans l'introduction pour $\vec{b} = (1, 0)$.

Dans le modèle de Groma-Balogh, les auteurs ont négligé les corrélations entre dislocation-dislocation et les effets de bords. Il est possible néanmoins de les prendre en compte, voir I. Groma, F. Csikor et M. Zaiser [72]. Plus précisément, ce modèle correspond à une généralisation du modèle de Groma-Balogh [71] et il s'agit d'un système couplé hyperbolique/parabolique. Pour une étude mathématique du modèle [72], H. Ibrahim a montré dans [78] l'existence et l'unicité d'une solution dans un cadre géométrique particulier unidimensionnel.

Nous remarquons aussi que le modèle de Groma-Balogh décrit la dynamique des densités de dislocations dans le cadre d'une seule direction de glissement. Pour une extension dans le cas de plusieurs directions de glissement, nous renvoyons à S. Yefimov [135, ch. 5.] et S. Yefimov, E. Van der Giessen [136]. Voir aussi chapitre 4 de la thèse pour une étude mathématique ainsi que pour une modélisation physique pour un modèle unidimensionnel des densités de dislocations avec plusieurs directions de glissement.

Concernant le cas $3D$ général, A. El Azab [46], M. Zaiser, T. Hochrainer [75,137,138], M. Koslowski et al. [91] et R. Monneau [108], se sont récemment intéressés à la modélisation de la dynamique des densités de dislocations dans l'espace tridimensionnel, mais il reste beaucoup de questions ouvertes pour établir une théorie tridimensionnlle de la dynamique des densités de dislocations.

Première partie

Modèles unidimensionnels

Chapitre 2

Existence et unicité de solution pour un système d'équations de type Burgers décrivant la dynamique de densités de dislocations

Ce chapitre est un article à paraître dans *SIAM Journal on Mathematical Analysis.*

Dans ce travail nous étudions un système unidimensionnel d'équations de type Burgers non-conservatives. Ce système provient de la modélisation de la dynamique des densités de dislocations dans les cristaux. Nous prouvons l'existence globale et l'unicité d'une solution dans la classe des fonctions $H^1_{loc}(\mathbb{R} \times [0, +\infty))$ croissantes. L'approche est faite en ajoutant un terme de viscosité sur le système, puis nous montrons une estimation d'énergie. Cette estimation assure assez de compacité au passage à la limite quand la viscosité disparaît. Un principe de comparaison est montré pour ce système ainsi qu'une application sur l'équation de Burgers classique.

Well-posedness theory for a non-conservative Burger-type system arising in dislocation dynamics

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Abstract

In this work we study a system of non-conservative Burgers type in one space dimension, arising in modeling the dynamics of dislocations densities in crystals. Starting from physically relevant initial data that are of a special form, namely non-decreasing, periodic plus linear functions, we prove the global existence and uniqueness of a solution in $H^1_{loc}(\mathbb{R} \times [0, +\infty))$ that preserves the nature of the initial data. The approach is made by adding some viscosity to the system, obtaining energy estimates and passing to the limit for vanishing viscosity. A comparison principle is shown for this system as well as an application in the case of classical Burgers equation.

AMS Classification : 35L45, 35Q53, 35Q72, 74H20, 74H25.
Keywords : System of Burgers equations, system of nonlinear transport equations, nonlinear hyperbolic system, dynamics of dislocations densities.

1 Introduction

1.1 Physical motivations and presentation of the model

Real crystals comprise certain defects in the organization of their crystalline structure called dislocations. In a particular case where these defects are parallel straight lines in the three-dimensional space, they can be viewed as points in a plan. Under the effect of exterior constraints, dislocations can move in a certain crystallographic direction called the slip direction. This slip direction is given by a vector called the "Burger's vector". The norm of this vector represents the amplitude of the generated deformation. (We refer to [74] for further physical explanation).

In this work, we are interested in the study of a 1-D sub-model of a problem

introduced by Groma and Balogh [71], initially proposed in the two-dimensional case. In fact, this 1-D sub-model was defined by El Hajj, Forcadel [48, Lemme 3.1].

This two-dimensional model is characterized by the fact that dislocations propagate in the plane (x_1, x_2) following two Burger's vectors $\pm\vec{b}$ with $\vec{b} = (1, 0)$. In this 1-D sub-model we suppose also that dislocations densities depend only on the variable $x = x_1 + x_2$, that transform the 2-D into a 1-D model (see El Hajj, Forcadel [48] for more modeling details).
More precisely this 1-D model is given by the following coupled equations of non-conservative Burgers type :

$$\begin{cases} \dfrac{\partial \rho^+}{\partial t}(x,t) &= -\left(a(t) + (\rho^+ - \rho^-)(x,t) + \alpha \displaystyle\int_0^1 (\rho^+ - \rho^-)(y,t)dy\right) \dfrac{\partial \rho^+}{\partial x}(x,t) \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times (0,T)), \\ \dfrac{\partial \rho^-}{\partial t}(x,t) &= \left(a(t) + (\rho^+ - \rho^-)(x,t) + \alpha \displaystyle\int_0^1 (\rho^+ - \rho^-)(y,t)dy\right) \dfrac{\partial \rho^-}{\partial x}(x,t) \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times (0,T)). \end{cases} \tag{1.1}$$

The unknowns ρ^+ and ρ^- are scalar valued functions, that we denote for simplicity by $\rho^\pm$. Their spatial derivatives $\dfrac{\partial \rho^\pm}{\partial x}$, are the dislocations densities of Burger's vector $\pm\vec{b} = \pm(1, 0)$. The function $a = a(t)$, representing the field of the imposed exterior constraint, is supposed to be independent of x, and the constant α depends on the elastic coefficients and the material size.
We consider the following initial conditions for (1.1) :

$$\rho^\pm(x, t = 0) = \rho_0^\pm(x) = \rho_0^{\pm,per}(x) + L_0 x \quad x \in \mathbb{R}, \tag{1.2}$$

where $\rho_0^{\pm,per}$ are 1-periodic functions. We thus modelize a periodic distribution for the $\pm$ dislocations, with a spatial period of length 1. Note that each type of $\pm$ dislocations have a mean density equal to L_0. In fact, the use of the periodic boundary conditions is a way of regarding what is going on in the interior of the material away from its boundary.

1.2 A brief review of some related literature

From a mathematical point of view, system (1.1) is related to other similar models, such as transport equations based on vector fields with low regularity. Such equations were for instance studied by Diperna, Lions in [45]. They proved the existence and uniqueness of a solution (in the renormalized sense), for vector fields in $L^1((0, +\infty); W^{1,1}_{loc}(\mathbb{R}^N))$ whose divergence are in $L^1((0, +\infty); L^\infty(\mathbb{R}^N))$. This study was generalized by Ambrosio [8], who considered vector fields in $L^1((0, +\infty); BV_{loc}(\mathbb{R}^N))$ with bounded divergence. In the present paper, we work in dimension $N = 1$ and

prove the existence and uniqueness of solutions of the system (1.1)-(1.2) with a vector field (i.e. the velocity) only in $L^\infty((0,+\infty), H^1_{loc}(\mathbb{R}))$.
We also refer to the works of LeFloch and Liu [99, 100] in which they considered the study in the framework of functions of bounded variation for a system of the form :

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + A(u)\dfrac{\partial u}{\partial x}(x,t) = 0 & u(x,t) \in U,\ x \in \mathbb{R},\ t \in (0,T), \\ \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{1.3}$$

where the space of states U is an open subset of $\mathbb{R}^p$, and A is $(p \times p)$ matrix which of class C^1 on U. Moreover $A(u)$ have p scalar distinct eigenvalues that we denote by : $\lambda_1(u) < \lambda_2(u) < ... < \lambda_p(u)$. We remark that this condition on the eigenvalues does not enter in our framework even in the case where $\alpha = a = 0$, because we have not sign property on $\rho^+ - \rho^-$. LeFloch and Liu proved that if the initial condition u_0 is sufficiently close to a constant state, and if the total variation $TV(u_0)$ is assumed to be small enough, then system (1.3) admits a solution in $L^\infty(\mathbb{R} \times (0,+\infty)) \cap BV(\mathbb{R} \times (0,+\infty))$, in the sense of weak entropy solutions with respect to admissible function (see LeFloch [99, Definition 3.2]).
When the system is hyperbolic and symmetric, this corresponds to the case $\alpha = a = 0$ in our system (1.1), it is proved in Serre [123, Vol I, Th 3.6.1] a result of local existence and uniqueness in $C([0,T); H^s(\mathbb{R}^N)) \cap C^1([0,T); H^{s-1}(\mathbb{R}^N))$, with $s > \frac{N}{2} + 1$, this result being only local in time, even in dimension $N = 1$.
The assumptions of increasing initial conditions was also considered in the study of the Euler equation for compressible fluids in dimension one. With regard to these studies, we refer to Chen and Wang [31, Th 3.1] for an existence and uniqueness result in $C^1(\mathbb{R} \times [0,+\infty))$ based on the method of characteristic. The result of Chen and Wang shows that the Euler equation of compressible fluids does not create shocks, for suitable increasing and $C^1(\mathbb{R})$ initial conditions. In our case, we already knew that solutions of (1.1), are Lipschitz continuous, see El Hajj and Forcadel [48]. Even if this regularity question is not concerned in the present paper, we may expect some $C^1(\mathbb{R} \times [0,+\infty))$ regularity of the solution for $C^1(\mathbb{R})$ initial data.

1.3 Main result

The main result of this paper is the existence and uniqueness of global in time solutions for the system (1.1)-(1.2), modeling the dynamics of dislocations densities. This result ensures the mathematical well-posedness of the Groma-Balogh model [71] in the particular case of our interest.

Theorem 1.1 (Existence and uniqueness)
For all $T, L_0 \geq 0$, $\alpha \in \mathbb{R}$ and $\rho_0^\pm \in H^1_{loc}(\mathbb{R})$ and under the following assumptions :

(H1) $\rho_0^{\pm}(x+1) = \rho_0^{\pm}(x) + L_0$, *(1-periodic function + linear function)*

(H2) $\dfrac{\partial \rho_0^{\pm}}{\partial x} \geq 0$, *a.e. in* $\mathbb{R}$, *(*$\rho_0^{\pm}$ *non-decreasing)*

(H3) $a \in L^{\infty}(0,T)$

the system (1.1)-(1.2) admits a unique solution $\rho^{\pm} \in H^1_{loc}(\mathbb{R} \times [0,T))$ *such that, for a.e.* $t \in (0,T)$, *the function* $\rho^{\pm}(.,t) : x \longmapsto \rho^{\pm}(x,t)$ *verifies* $(H1)$ *and* $(H2)$.

The preceding theorem gives a global existence and uniqueness result of the system (1.1). Its proof is based on the following steps. Firstly, we regularize the system (1.1), then we show a uniform *a priori* estimates in $L^{\infty}((0,T); H^1_{loc}(\mathbb{R}))$ for this regularized system. These estimates lead to a result of existence for long time solution and assure the passage to the limit by compactness. Finally, the demonstration of uniqueness is done in a direct way.

Theorem 1.2 (Comparison principle for (1.1) with $\alpha = 0$)
Let $a(\cdot)$ *satisfy* $(H3)$ *and* $\rho_1^{\pm,}$, $\rho_2^{\pm} \in H^1_{loc}(\mathbb{R} \times [0,T))$ *be two solutions of the system (1.1) with* $\alpha = 0$. *Moreover, let* $\rho_1^{\pm}(.,t)$, $\rho_2^{\pm}(.,t)$ *verify* $(H1)$ *and* $(H2)$ *for a.e.* $t \in (0,T)$. *Then, if* $\rho_1^{\pm}(\cdot,0) \leq \rho_2^{\pm}(\cdot,0)$ *in* $\mathbb{R}$, *we have* $\rho_1^{\pm} \leq \rho_2^{\pm}$ *a.e. in* $\mathbb{R} \times (0,T)$.

This comparison result was crucial in a previous work [48], for the demonstration of existence and uniqueness of Lipschitz solution to problem (1.1), in the sense of viscosity solution, for Lipschitz initial conditions. Here, the interest of this result is a little bit secondary. Indeed, thanks to this comparison principle, we have been able to obtain indirectly $H^1_{loc}(\mathbb{R} \times [0,T))$ estimates. These estimates in their turn lead to a result of existence in $H^1_{loc}(\mathbb{R} \times [0,T))$.
Our work focuses on the study of the dynamics of dislocations densities. In a different direction, let us quote some recent results on the dynamics of dislocations lines, taken individually, that are represented by non-local Hamilton-Jacobi equations (see [7,55] and [3, 15] for local and global in time results respectively).

Remark 1.3 (Existence and uniqueness for Burgers equation)
We remark that these technics can be applied to the case of classical Burgers equations in $W^{1,p}_{loc}(\mathbb{R} \times [0,T))$ *for all* $1 \leq p < +\infty$.
Indeed, if we consider for a given function f *and initial data* u_0, *the following equation :*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(f(u)) = 0 & \text{in} \quad \mathcal{D}'(\mathbb{R} \times (0,T)) \\ \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{1.4}$$

then we have the following theorem :

Theorem 1.4 *Let $T \geq 0$, $p \in [1, +\infty)$ and f locally lipschitz and convex. Then for all initial data $u_0 \in W^{1,p}_{loc}(\mathbb{R})$, such that $\dfrac{\partial u_0}{\partial x} \in L^p(\mathbb{R})$ and satisfies $(H1)$. The equation (1.4) admits a solution $u \in W^{1,p}_{loc}(\mathbb{R} \times [0, T))$, unique in the class of solutions satisfying $(H1)$ a.e. $t \in (0, T)$.*

1.4 Organization of the paper

In section 2, we regularize the function $a(\cdot)$ and the initial conditions and we prove that the system (1.1)-(1.2) modified by the term $(\varepsilon \dfrac{\partial^2 \rho^{\pm}}{\partial x^2})$ admits local in time solutions (in the "Mild" sense). This will be achieved by using an application of a fixed point theorem in the space of functions in $C([0, T); H^1_{loc}(\mathbb{R}))$ and verifying $(H1)$ for all $t \in (0, T)$. In section 3, we prove that the obtained solutions are regular and verify $(H2)$ for all $t \in (0, T)$, with initial conditions verifying $(H2)$. In section 4, we prove some uniform *a priori* estimates on the regularized solution obtained in section 3. Thanks to these estimates, we also prove the existence of global in time solutions. In section 5, we give the demonstration of Theorem 1.1 and in section 6 we prove a comparison principle result of the system (1.1) in the case $\alpha = 0$. Finally, in section 7 we give an application of the previous results in the case of the classical Burgers equation.

2 Existence of solutions for an approximated system

In this paragraph, we prove a theorem of existence of solutions, local in time, for the system (1.1) modified by the term $\varepsilon \dfrac{\partial^2 \rho^{\pm}}{\partial x^2}$ after the regularization of the function $a(\cdot)$ and the initial conditions. This approximation brings us back to the study, for every $0 < \varepsilon < 1$, of the following system :

$$\begin{cases} \dfrac{\partial \rho^{+,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 \rho^{+,\varepsilon}}{\partial x^2} = & -\left(a^{\varepsilon}(t) + (\rho^{+,\varepsilon} - \rho^{-,\varepsilon}) + \alpha \displaystyle\int_0^1 (\rho^{+,\varepsilon} - \rho^{-,\varepsilon})(y,t)dy \right) \dfrac{\partial \rho^{+,\varepsilon}}{\partial x} \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times (0,T)), \\ \dfrac{\partial \rho^{-,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 \rho^{-,\varepsilon}}{\partial x^2} = & \left(a^{\varepsilon}(t) + (\rho^{+,\varepsilon} - \rho^{-,\varepsilon}) + \alpha \displaystyle\int_0^1 (\rho^{+,\varepsilon} - \rho^{-,\varepsilon})(y,t)dy \right) \dfrac{\partial \rho^{-,\varepsilon}}{\partial x} \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times (0,T)), \end{cases} \tag{2.5}$$

where $a^{\varepsilon} = \tilde{a} * \eta_{\varepsilon}$,with $\eta_{\varepsilon}(\cdot) = \frac{1}{\varepsilon}\eta(\frac{\cdot}{\varepsilon})$, such that $\eta \in C^{\infty}_c(\mathbb{R})$, positive, and $\int_{\mathbb{R}} \eta = 1$. The function $\tilde{a}(\cdot)$ is an extension in $\mathbb{R}$ of the function $a(\cdot)$ by 0.

We also consider the regularized initial conditions of the system (2.5) :

$$\rho^{\pm,\varepsilon}(x,0) = \rho_0^{\pm,\varepsilon}(x) = \rho_0^{\pm,\varepsilon,per}(x) + L_0 x = \rho_0^{\pm,per} *_{\mathbb{T}} \eta_\varepsilon(x) + L_0 x, \tag{2.6}$$

We have the following local in time existence result for the approximated system :

Theorem 2.1 (Short time existence) *Assume* $(H1)$ *and* $(H3)$. *For all* $\alpha \in \mathbb{R}$ *and* $\rho_0^{\pm} \in H^1_{loc}(\mathbb{R})$ *there exists*

$$T^\star(\|\rho_0^{\pm,per}\|_{H^1(\mathbb{T})}, \|a\|_{L^\infty(0,T)}, L_0, \alpha, \varepsilon) > 0,$$

such that the system (2.5)-(2.6) admits a solution $\rho^{\pm,\varepsilon} \in C([0,T^\star); H^1_{loc}(\mathbb{R}))$ *with* $\rho^{\pm,\varepsilon}(.,t)$ *verifying* $(H1)$.

For the proof of this theorem (see sub-section 2.3). Before going on, we need to give some notation and preliminary results that will be used througout the paper.

2.1 Notation

In what follows, we are going to use the following notation :

1. $\rho^\varepsilon = \rho^{+,\varepsilon} - \rho^{-,\varepsilon}$,
2. $\rho^{\pm,\varepsilon,per} = \rho^{\pm,\varepsilon} - L_0 x$,
3. $\mathbb{T} = (\mathbb{R}/\mathbb{Z})$ is the $[0,1)$ periodic interval,
4. let $f = (f_1, f_2)$ be a vector such that $f_i \in H^1(\mathbb{T})$ for $i \in \{1,2\}$. The norm of f in $(H^1(\mathbb{T}))^2$ will be defined by $\|f\|_{H^1(\mathbb{T})} = \max(\|f_1\|_{H^1(\mathbb{T})}, \|f_2\|_{H^1(\mathbb{T})})$.
5. Let f be a function from $\mathbb{R} \times (0,T)$ to $\mathbb{R}$. we note by $f(t) = f(.,t) : x \longmapsto f(x,t)$.

Remark 2.2 (Periodicity) *According to* $(H1)$-$(H2)$, *it is clear that* ρ^ε, $\rho^{\pm,\varepsilon,per}$ *and* $\dfrac{\partial \rho^{\pm,\varepsilon}}{\partial x}$ *are* 1*-periodic in space functions.*

Under the notation of paragraph 2.1, we know that the system (2.5) is equivalent to :

$$\frac{\partial \rho^{\pm,\varepsilon,per}}{\partial t} - \varepsilon \frac{\partial^2 \rho^{\pm,\varepsilon,per}}{\partial x^2} = \mp \overbrace{C_\alpha[\rho^\varepsilon(t)] \frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x}}^{\text{bilinear term}} \mp \overbrace{a^\varepsilon(t) \frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x} \mp L_0 C_\alpha[\rho^\varepsilon(t)]}^{\text{linear term}} \mp L_0 a^\varepsilon(t) \quad \text{in } \mathbb{T}\times(0,T), \tag{2.7}$$

where $C_\alpha[\rho^\varepsilon(t)](x) = \left(\rho^\varepsilon(x,t) + \alpha \displaystyle\int_0^1 \rho^\varepsilon(y,t)dy\right)$,

with the periodic initial conditions

$$\rho^{\pm,\varepsilon,per}(x,0) = \rho_0^{\pm,\varepsilon,per}(x) \quad \text{in} \quad \mathbb{T}. \tag{2.8}$$

2.2 Preliminary results

***Lemma* 2.3 (Properties of the regularized sequence)** *Under the hypothesis* $(H1)$ *and* $(H3)$ *and for every* $\rho_0^{\pm} \in H^1_{loc}(\mathbb{R})$, *we have*

1. *The functions* $\rho_0^{\pm,\varepsilon,per} \in C^\infty(\mathbb{T})$ *and verify the following estimate :*
$$\|\rho_0^{\pm,\varepsilon,per}\|_{H^1(\mathbb{T})} \leq C\|\rho_0^{\pm,per}\|_{H^1(\mathbb{T})}.$$
2. *The function* $a^\varepsilon(\cdot) \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *and verifies the following estimate :*
$$\|a^\varepsilon\|_{L^\infty(\mathbb{R})} \leq \|a\|_{L^\infty(0,T)}.$$
3. *The sequence* $a^\varepsilon(\cdot)$ *strongly converges to* $a(\cdot)$ *in* $L^2(0,T)$. *The sequences* $\rho_0^{\pm,\varepsilon,per}$ *strongly converge to* $\rho_0^{\pm,per}$ *in* $H^1(\mathbb{T})$.

The proof of this lemma is a classical property of the regularizing sequence $(\eta_\varepsilon)_\varepsilon$.

***Lemma* 2.4 (Mild solution)** *Assume* $(H3)$. *For every* $T \geq 0$, *if* $\rho^{\pm,\varepsilon,per} \in C([0,T); H^1(\mathbb{T}))$ *are solutions of the following equation :*

$$\begin{aligned}\rho^{\pm,\varepsilon,per}(x,t) = S_\varepsilon(t)\rho_0^{\pm,\varepsilon,per} \quad & \mp L_0 \int_0^t a^\varepsilon(s)ds \mp \int_0^t S_\varepsilon(t-s)\left(C_\alpha[\rho^\varepsilon(s)]\frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x}(s)\right)ds \\ & \mp \int_0^t S_\varepsilon(t-s)\left(L_0 C_\alpha[\rho^\varepsilon(s)] + a(t)\frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x}(s)\right)ds,\end{aligned} \tag{2.9}$$

where $S_\varepsilon(t) = e^{\varepsilon t\Delta}$ *is the heat semi-group, then* $\rho^{\pm,\varepsilon,per}$ *is a solution of the system (2.7)-(2.8) in the sense of distributions.*

For the proof of this lemma, we refer to Pazy [117, Th 5.2. Page 146].

***Lemma* 2.5 (Fixed point)** *Let* E *be a Banach space,* B *be a continuous bilinear application from* $E \times E$ *to* E *and* L *be a continuous linear application from* E *to* E *such that :*

$$\|B(x,y)\|_E \leq \lambda\|x\|_E\|y\|_E \quad \textit{for all} \quad x,y \in E,$$
$$\|L(x)\|_E \leq \mu\|x\|_E \quad \textit{for all} \quad x \in E,$$

where $\lambda > 0$ *and* $\mu \in (0,1)$ *are given constants. Then, for all* $x_0 \in E$ *such that*

$$\|x_0\|_E < \frac{1}{4\lambda}(\mu-1)^2,$$

the equation $x = x_0 + B(x,x) + L(x)$ *admits a solution in* E.

For the proof of this lemma we refer to Cannone [24, Lemma 4.2.14].

In order to show the existence of a solution within the framework of Lemma 2.4, we apply Lemma 2.5 in the space $E = (L^\infty((0,T);H^1(\mathbb{T})))^2$, where x_0, B and L are defined, for $u = (u_1, u_2)$, $v = (v_1, v_2) \in E$, by :

$$x_0 = S_\varepsilon(t)\rho^\varepsilon_{0,vec} + L_0\vec{i}\int_0^t a^\varepsilon(s)ds, \quad \text{where} \quad \rho^\varepsilon_{0,vec} = (\rho_0^{+,\varepsilon,per}, \rho_0^{-,\varepsilon,per}),\ \vec{i} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{2.10}$$

$$B(u,v)(t) = \bar{I}_1 \int_0^t S_\varepsilon(t-s)\left(C_\alpha[u_1(s) - u_2(s)]\frac{\partial v}{\partial x}(s)\right)ds, \quad \text{where} \quad \bar{I}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.11}$$

$$L(u)(t) = L_0\vec{i}\int_0^t S_\varepsilon(t-s)C_\alpha[u_1(s) - u_2(s)]ds + \bar{I}_1\int_0^t S_\varepsilon(t-s)\left(a^\varepsilon(s)\frac{\partial u}{\partial x}(s)\right)ds. \tag{2.12}$$

***Lemma* 2.6 (Decreasing estimates)** *If $f \in L^q(\mathbb{T})$ with $2 \le q \le +\infty$ and $g \in L^2(\mathbb{T})$, then for all $t > 0$ we have the following estimates :*

(i)

$$\|S_\varepsilon(t)(fg)\|_{L^\infty(\mathbb{T})} \le Ct^{-\frac{1}{2}}\|f\|_{L^2(\mathbb{T})}\|g\|_{L^2(\mathbb{T})},$$

(ii)

$$\left\|\frac{\partial}{\partial x}(S_\varepsilon(t)f)\right\|_{L^2(\mathbb{T})} \le Ct^{-\frac{1}{2}}\|S_\varepsilon(\frac{t}{2})f\|_{L^2(\mathbb{T})},$$

(iii)

$$\left\|\frac{\partial}{\partial x}\left(S_\varepsilon(t)(fg)\right)\right\|_{L^2(\mathbb{T})} \le Ct^{-\frac{1}{2}(1+\frac{1}{q})}\|f\|_{L^q(\mathbb{T})}\|g\|_{L^2(\mathbb{T})},$$

where $C = C(\varepsilon)$ is a positive constant depending on ε.

For the proof of this lemma, see Pazy [117, Lemma 1.1.8, Th 6.4.5].

Proposition 2.7 (Bilinear operator) *Let $F_T = (L^\infty((0,T);H^1(\mathbb{T})))^2$. Then for every $T \ge 0$, $\alpha \in \mathbb{R}$, $u = (u_1, u_2) \in F_T$ and $v = (v_1, v_2) \in F_T$ the bilinear operator B defined in (2.11), is continuous from $F_T \times F_T$ to F_T. Moreover, there exists a positive constant $C = C(\alpha, \varepsilon)$ such that for all $u, v \in F_T$ we have :*

$$\|B(u,v)\|_{F_T} \le CT^{\frac{1}{2}}\|u\|_{F_T}\|v\|_{F_T}.$$

Proof of Proposition 2.7
Firstly, we know

$$\begin{aligned}\|B(u,v)(t)\|_{H^1(\mathbb{T})} &= \left\|\bar{I}_1\int_0^t S_\varepsilon(t-s)\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial v}{\partial x}(s)\right)ds\right\|_{H^1(\mathbb{T})}\\ &\leq \int_0^t \left\|S_\varepsilon(t-s)\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial v}{\partial x}(s)\right)\right\|_{H^1(\mathbb{T})}ds\end{aligned}$$

Then, since $L^\infty(\mathbb{T}) \hookrightarrow L^2(\mathbb{T})$, we have

$$\begin{aligned}\|B(u,v)(t)\|_{H^1(\mathbb{T})} &\leq \int_0^t \left\|S_\varepsilon(t-s)\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial v}{\partial x}(s)\right)\right\|_{L^\infty(\mathbb{T})}ds\\ &+\int_0^t \left\|\frac{\partial}{\partial x}S_\varepsilon(t-s)\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial v}{\partial x}(s)\right)\right\|_{L^2(\mathbb{T})}ds.\end{aligned}$$

Using Lemma 2.6 (i) for the first term and Lemma 2.6 (iii) with $q=\infty$ for the second term, we can conclude that :

$$\begin{aligned}\|B(u,v)(t)\|_{H^1(\mathbb{T})} &\leq C\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}}\|C_\alpha[u_1(s)-u_2(s)]\|_{L^\infty(\mathbb{T})}\left\|\frac{\partial v}{\partial x}(s)\right\|_{L^2(\mathbb{T})}ds\\ &\leq C\sup_{0\leq t<T}(\|u(t)\|_{H^1(\mathbb{T})})\sup_{0\leq t<T}(\|v(t)\|_{H^1(\mathbb{T})})\int_0^t\frac{1}{(t-s)^{\frac{1}{2}}}ds.\end{aligned}$$

Then for all $t\in(0,T)$, we have :

$$\begin{aligned}\|B(u,v)(t)\|_{H^1(\mathbb{T})} &\leq Ct^{\frac{1}{2}}\|u\|_{L^\infty((0,T);H^1(\mathbb{T}))^2}\|v\|_{L^\infty((0,T);H^1(\mathbb{T}))^2}\\ &\leq CT^{\frac{1}{2}}\|u\|_{L^\infty((0,T);H^1(\mathbb{T}))^2}\|v\|_{L^\infty((0,T);H^1(\mathbb{T}))^2}.\end{aligned} \tag{2.13}$$

□

Proposition 2.8 (Linear operator) *Let $F_T=(L^\infty((0,T);H^1(\mathbb{T})))^2$ and $a(\cdot)$ satisfying $(H3)$. Then for all $L_0,\ T\geq 0$ and $u=(u_1,u_2)\in F_T$, the linear operator L defined in (2.12), is continuous from F_T to F_T. Moreover, there exists a positive constant $C=C(\alpha,\varepsilon,\|a\|_{L^\infty(0,T)},L_0)$ such that :*

$$\|L(u)\|_{F_T}\leq CT^{\frac{1}{2}}\|u\|_{F_T}.$$

The proof of Proposition 2.8 is similar of the one used in Proposition 2.7.

***Lemma* 2.9** *For all $L_0,\ T\geq 0$ and $a(\cdot)$ satisfying $(H3)$, if*

$$X_{a^\varepsilon}(t)=L_0\vec{i}\int_0^t a^\varepsilon(s)ds,\ t\in(0,T),$$

then

$$\|X_{a^\varepsilon}\|_{(L^\infty(0,T))^2} \leq L_0 T \|a\|_{L^\infty(0,T)}.$$

The proof of Lemma 2.9 is trivial (from Lemma 2.3 (2)).

***Lemma* 2.10 (Continuity of the semi-group)** *For all $f \in W^{2,2}(\mathbb{T})$ and $0 \leq \theta < t$, we have the following estimates :*

(i)

$$\|(S_\varepsilon(t-\theta) - Id)f\|_{L^2(\mathbb{T})} \leq C(t-\theta)\|\frac{\partial^2 f}{\partial x^2}\|_{L^2(\mathbb{T})},$$

(ii)

$$\|(S_\varepsilon(t-\theta) - Id)f\|_{L^2(\mathbb{T})} \leq 2\|f\|_{L^2(\mathbb{T})},$$

where $C = C(\varepsilon)$ is a positive constant depending on ε.

We refer to Pazy [117, Lemma 6.2 Page 151] for the proof of this lemma.

***Lemma* 2.11 (Time continuity)** *Assume $(H3)$. If $\rho_{0,vec} = (\rho_0^{+,per}, \rho_0^{-,per}) \in (H^1(\mathbb{T}))^2$, then for all $T \geq 0$ and $u = (u_1, u_2) \in (L^\infty((0,T); H^1(\mathbb{T})))^2$, the following applications :*

(A1) : $t \to X_{a^\varepsilon}(t)$,

(A2) : $t \to S_\varepsilon(t)\rho^\varepsilon_{0,vec}$, where $\rho^\varepsilon_{0,vec} = (\rho_0^{+,\varepsilon,per}, \rho_0^{-,\varepsilon,per})$,

(A3) : $t \to B(u,u)(t)$,

(A4) : $t \to L(u)(t)$,

are $(C([0,T); H^1(\mathbb{T})))^2$. Where X_{a^ε}, B and L are defined in Lemma 2.9, (2.11) and (2.12) respectively.

Proof of Lemma 2.11

The continuity of $(A1)$ is trivial since $a \in L^\infty(0,T)$. From the fact that the semi-group $S_\varepsilon(\cdot)$ is continuous from $[0,T)$ to $(H^1(\mathbb{T}))^2$ we deduce the continuity of $(A2)$. It remains to prove the continuity of $(A3)$ and $(A4)$. Indeed, the continuity of $(A3)$ at 0 is a consequence of inequality (2.13). Now, we are going to prove the continuity of $(A3)$ for all $\theta \in (0,T)$. For all t, such that $\theta < t \leq \min(T, \frac{3\theta}{2})$, we write $t = (1+\gamma)\theta$ and denote $\tau = (1-\gamma)\theta$ (where $0 < \gamma \leq \frac{1}{2}$) and we write

$$\begin{aligned} B(u,u)(t) - B(u,u)(\theta) &= \int_0^\tau (S(t-s) - S(\theta-s)) \left(C_\alpha[u_1(s) - u_2(s)]\frac{\partial u}{\partial x}(s) \right) ds \\ &+ \int_\tau^\theta (S(t-s) - S(\theta-s)) \left(C_\alpha[u_1(s) - u_2(s)]\frac{\partial u}{\partial x}(s) \right) ds \\ &+ \int_\theta^t S(t-s) \left(C_\alpha[u_1(s) - u_2(s)]\frac{\partial u}{\partial x}(s) \right) ds \end{aligned}$$

$$= \overbrace{\int_0^\tau ((S(t-\theta)-Id)S(\theta-s))\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial u}{\partial x}(s)\right)ds}^{I_1}$$

$$+\overbrace{\int_\tau^\theta ((S(t-\theta)-Id)S(\theta-s))\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial u}{\partial x}(s)\right)ds}^{I_2}$$

$$+\int_\theta^t S(t-s)\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial u}{\partial x}(s)\right)ds.$$

We apply Lemma 2.10 (i) and Lemma 2.6 (ii) to find an upper bound to I_1. We then apply Lemma 2.10 (ii) to find an upper bound to I_2. After that, we follow the same steps of the proof of Proposition 2.7 to conclude that :

$$\begin{aligned}\|B(u,u)(t)-B(u,u)(\theta)\|_{H^1} \leq\ & C(t-\theta)\|u\|^2_{(L^\infty((0,T);H^1(\mathbb{T})))^2}\int_0^\tau \frac{1}{(\theta-s)^{\frac{3}{2}}}ds \\ & +C\|u\|^2_{(L^\infty((0,T);H^1(\mathbb{T})))^2}\int_\tau^\theta \frac{1}{(\theta-s)^{\frac{1}{2}}}ds \\ & +C\|u\|^2_{(L^\infty((0,T);H^1(\mathbb{T})))^2}\int_\theta^t \frac{1}{(t-s)^{\frac{1}{2}}}ds.\end{aligned}$$

After the computation of each integral we deduce that :

$$\begin{aligned}\|B(u,u)(t)-B(u,u)(\theta)\|_{H^1} \leq\ & C(t-\theta)\left(\tfrac{1}{(\theta-\tau)^{\frac{1}{2}}}-\tfrac{1}{\theta^{\frac{1}{2}}}\right)\|u\|^2_{(L^\infty((0,T);H^1(\mathbb{T})))^2} \\ & +C\left((\theta-\tau)^{\frac{1}{2}}+(t-\theta)^{\frac{1}{2}}\right)\|u\|^2_{(L^\infty((0,T);H^1(\mathbb{T})))^2}.\end{aligned}$$

Observing that $t-\theta=\theta-\tau=\gamma\theta$ we finally obtain the following inequality :

$$\|B(u,u)(t)-B(u,u)(\theta)\|_{H^1} \leq C(\theta,\gamma)\left((t-\theta)^{\frac{1}{2}}+(t-\theta)\right)\|u\|^2_{(L^\infty((0,T);H^1(\mathbb{T})))^2},$$

hence the continuity of $(A3)$. In the same way we get the continuity in time of $(A4)$. □

2.3 Proof of Theorem 2.1

Proof of Theorem 2.1
We rewrite the system (2.9) in the following vectorial form :

$$\begin{aligned}\rho^{\varepsilon}_{vec}(\cdot,t) = \; & S_{\varepsilon}(t)\rho^{\varepsilon}_{0,vec} + L_0\vec{i}\int_0^t a^{\varepsilon}(s)ds + \bar{I}_1\int_0^t S_{\varepsilon}(t-s)\left(C_{\alpha}[\rho^{\varepsilon}(s)]\frac{\partial\rho^{\varepsilon}_{vec}}{\partial x}(s)\right)ds \\ & + L_0\vec{i}\int_0^t S_{\varepsilon}(t-s)C_{\alpha}[\rho^{\varepsilon}(s)]ds + \bar{I}_1\int_0^t S_{\varepsilon}(t-s)\left(a^{\varepsilon}(s)\frac{\partial\rho^{\varepsilon}_{vec}}{\partial x}(s)\right)ds.\end{aligned}$$

Such that ρ^{ε}_{vec} is the vector $(\rho^{+,\varepsilon,per},\rho^{-,\varepsilon,per})$ and $\rho^{\varepsilon}_{0,vec}$ is the vector $(\rho_0^{+,\varepsilon,per},\rho_0^{-,\varepsilon,per})$.
$\vec{i}$ and $\bar{I}_1$ are defined in (2.10) and (2.11) respectively.
This altogether leads to the following equation :

$$\rho^{\varepsilon}_{vec}(\cdot,t) = S_{\varepsilon}(t)\rho^{\varepsilon}_{0,vec} + X_{a^{\varepsilon}}(t) + B(\rho^{\varepsilon}_{vec},\rho^{\varepsilon}_{vec})(t) + L(\rho^{\varepsilon}_{vec})(t), \tag{2.14}$$

where B is the bilinear application and L is the linear application defined in (2.11) and (2.12) respectively and $X_{a^{\varepsilon}}$ is defined in Lemma 2.9. Moreover, according to Lemmas 2.9 and 2.3 we know that :

$$\begin{aligned}\|S(t)\rho^{\varepsilon}_{0,vec} + X_{a^{\varepsilon}}(t)\|_{(L^{\infty}((0,T);H^1(\mathbb{T})))^2} & \leq \|\rho^{\varepsilon}_{0,vec}\|_{H^1(\mathbb{T})} + L_0T\|a^{\varepsilon}\|_{L^{\infty}(\mathbb{R})} \\ & \leq C_0|\rho_{0,vec}\|_{H^1(\mathbb{T})} + L_0T\|a\|_{L^{\infty}(0,T)}.\end{aligned}$$

In order to apply Lemma 2.5, we want, for a well chosen time T, that the following inequality holds :

$$C_0\|\rho^0_{vec}\|_{H^1(\mathbb{T})} + L_0T\|a\|_{L^{\infty}(0,T)} < \frac{1}{4CT^{\frac{1}{2}}}(CT^{\frac{1}{2}}-1)^2, \text{ and } CT^{\frac{1}{2}} < 1, \tag{2.15}$$

where C is the largest constant between the two constants computed in Propositions 2.8 and 2.7. For :

$$(T^{\star})^{\frac{1}{2}}(\|\rho_{0,vec}\|_{H^1(\mathbb{T})},\|a\|_{L^{\infty}(0,T)},L_0,\varepsilon) = \min\left(1,\frac{1}{2C},\frac{1}{16C(C_0\|\rho^0_{vec}\|_{H^1(\mathbb{T})} + L_0\|a\|_{L^{\infty}(0,T)})}\right), \tag{2.16}$$

we can easily verify that $T^{\star}$ satisfies the inequality (2.15). We apply Lemma 2.5 over the space $F_{T^{\star}} = (L^{\infty}((0,T^{\star});H^1(\mathbb{T})))^2$, to prove the existence of a solution for the system (2.14) in $F_{T^{\star}}$.
Then, according to Lemma 2.11, we deduce that the obtained solution is $(C([0,T^{\star});H^1(\mathbb{T})))^2$.
This proves, by Lemma 2.4, the existence of a solution in the sense of distributions for the system (2.5)-(2.6) in $C([0,T^{\star});H^1_{loc}(\mathbb{R}))$ that verifies $(H1)$. □

3 Properties of the solution to the approximated system

In this section we show that, the solutions of system (2.5)-(2.6) obtained in the previous section, are regular and verify $(H2)$, provided with initial conditions verify $(H2)$.

***Lemma* 3.1 (Regularity of the solution)** *Assume* $(H1)$, $(H3)$ *and* $\rho_0^{\pm} \in H^1_{loc}(\mathbb{R})$, *if* $\rho^{\pm,\varepsilon} \in C([0,T); H^1_{loc}(\mathbb{R}))$ *are solutions of the system (2.5)-(2.6), then* $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R} \times [0,T))$.

Proof of Lemma 3.1
If we denote the second term of the system (2.7) by

$$f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon(t)] = \mp a^\varepsilon(t)\left(L_0 + \frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x}\right) \mp C_\alpha[\rho^\varepsilon(t)]\left(\frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x} + L_0\right),$$

we know that $f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon] \in L^2(\mathbb{T}\times(0,T))$. Moreover, we know that the initial conditions $\rho_0^{\pm,\varepsilon,per} \in C^\infty(\mathbb{T})$, which allows us to apply the L^2 regularity of the heat equation over the system (2.7)-(2.8) (see Lions-Magenes [105, Th.8.2]). Then we deduce by induction that the solution is $C^\infty(\mathbb{T} \times [0,T))$. □

***Lemma* 3.2 (Monotonicity of the solution in space)** *Assume* $(H1)$, $(H2)$, $(H3)$ *and* $\rho_0^{\pm} \in H^1_{loc}(\mathbb{R})$, *if* $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R} \times [0,T))$ *are solutions of the system (2.5)-(2.6), then* $\rho^{\pm,\varepsilon}(.,t)$ *verifies* $(H2)$ *for all* $t \in (0,T)$.

Proof of Lemma 3.2
First, we remark that if $\dfrac{\partial \rho_0^{\pm}}{\partial x} \geq 0$, then $\dfrac{\partial \rho_0^{\pm,\varepsilon}}{\partial x} \geq 0$. Indeed, we have

$$\begin{aligned}\frac{\partial \rho_0^{\pm,\varepsilon}}{\partial x} = \frac{\partial \rho_0^{\pm,per}}{\partial x} * \eta_\varepsilon + L_0 &= \left(\frac{\partial \rho_0^{\pm,per}}{\partial x} + L_0\right) * \eta_\varepsilon \\ &= \left(\frac{\partial \rho_0^{\pm}}{\partial x}\right) * \eta_\varepsilon \geq 0, \quad \text{because } \eta \text{ is positive.}\end{aligned}$$

We apply the maximum principle over the derived system of (2.5)-(2.6) :

$$\begin{cases} \dfrac{\partial \theta^{\pm,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 \theta^{\pm,\varepsilon}}{\partial x^2} \pm (C_\alpha[\rho^\varepsilon(t)] + a^\varepsilon(t))\dfrac{\partial \theta^{\pm,\varepsilon}}{\partial x} \pm (\theta^{+,\varepsilon} - \theta^{-,\varepsilon})\theta^{\pm,\varepsilon} = 0 \quad \text{in} \quad \mathbb{T} \times (0,T), \\ \theta^{\pm,\varepsilon}(x,0) = \dfrac{\partial \rho_0^{\pm,\varepsilon}}{\partial x}, \end{cases},$$

where $\theta^{\pm,\varepsilon} = \dfrac{\partial \rho^{\pm,\varepsilon}}{\partial x}$ (see Gilbarg-Trudinger [63, Th.8.1])). Since $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R} \times [0,T))$, we deduce that $\theta^{\pm,\varepsilon} \geq 0$ belongs to $\mathbb{T} \times (0,T)$. □

Corollary 3.3 (Short time existence of non-decreasing regular solutions) *For all $\alpha \in \mathbb{R}$ and $\rho_0^{\pm} \in H^1_{loc}(\mathbb{R})$, under the assumptions $(H1)$, $(H2)$ and $(H3)$, there exists*

$$T^\star(\|\rho_0^{\pm,per}\|_{H^1(\mathbb{T})}, \|a\|_{L^\infty(0,T)}, L_0, \alpha, \varepsilon) > 0,$$

such that the system (2.5)-(2.6) admits a solution $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R} \times [0,T^\star))$ with $\rho^{\pm,\varepsilon}(.,t)$ verifying $(H1)$ and $(H2)$.

The proof of Corollary 3.3 is a consequence of Theorem 2.1 and Lemmas 3.1 and 3.2 (with $T = T^\star$).

Remark 3.4 *Here, we remark that the case of non-decreasing solutions corresponds to a non-shock case in Burgers equation. On the other hand, the decreasing solutions represent the shock case.*

4 *A priori* estimates and long time existence for the approximated system

In this paragraph, we are going to show some ε-uniform estimates on the solutions of the system (2.5)-(2.6). These estimates will be used in section 4 for the passage to the limit as ε tends to zero.

***Lemma* 4.1 (L^2 estimates over the space derivatives of the solutions)** *Assume $(H1)$, $(H2)$, $(H3)$ and $\rho_0^{\pm} \in H^1_{loc}(\mathbb{R})$, if $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R} \times [0,T))$ is a solution of the system (2.5)-(2.6) for all $T \geq 0$, then*

$$\left\|\frac{\partial \rho^{+,\varepsilon}}{\partial x}\right\|^2_{L^\infty((0,T);L^2(\mathbb{T}))} + \left\|\frac{\partial \rho^{-,\varepsilon}}{\partial x}\right\|^2_{L^\infty((0,T);L^2(\mathbb{T}))} \leq CB_0,$$

with $B_0 = \left(\left\|\dfrac{\partial \rho_0^+}{\partial x}\right\|^2_{L^2(\mathbb{T})} + \left\|\dfrac{\partial \rho_0^-}{\partial x}\right\|^2_{L^2(\mathbb{T})}\right)$.

Proof of Lemma 4.1

If we denote $\rho^\varepsilon = \rho^{+,\varepsilon} - \rho^{-,\varepsilon}$ and $k^\varepsilon = \rho^{+,\varepsilon} + \rho^{-,\varepsilon}$ then, according to $(H1)$, it is clear that ρ^ε, $\dfrac{\partial \rho^\varepsilon}{\partial x}$ and $\dfrac{\partial k^\varepsilon}{\partial x}$ are 1-periodic functions. Moreover, by Lemma 3.2, we know that $\dfrac{\partial k^\varepsilon}{\partial x} \geq 0$.

If we take into consideration the equations of the system (2.5), we can conclude that ρ^ε and k^ε verify the following system :

$$\begin{cases} \dfrac{\partial \rho^\varepsilon}{\partial t} - \varepsilon \dfrac{\partial^2 \rho^\varepsilon}{\partial x^2} &= -\left(\rho^\varepsilon + \alpha \displaystyle\int_0^1 \rho^\varepsilon dx + a^\varepsilon(t)\right) \dfrac{\partial k^\varepsilon}{\partial x} \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times (0,T)), \\ \dfrac{\partial k^\varepsilon}{\partial t} - \varepsilon \dfrac{\partial^2 k^\varepsilon}{\partial x^2} &= -\left(\rho^\varepsilon + \alpha \displaystyle\int_0^1 \rho^\varepsilon dx + a^\varepsilon(t)\right) \dfrac{\partial \rho^\varepsilon}{\partial x} \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times (0,T)). \end{cases} \tag{4.17}$$

We derive the first equation of the system (4.17) with respect to x, then we multiply the result by $\dfrac{\partial \rho^\varepsilon}{\partial x}$ and finally we integrate in space. For all $t \in (0,T)$, we then obtain :

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\|\frac{\partial \rho^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} + \varepsilon \left\|\frac{\partial^2 \rho^\varepsilon}{\partial x^2}(t)\right\|^2_{L^2(\mathbb{T})} = & -\int_0^1 (\frac{\partial \rho^\varepsilon}{\partial x})^2 \frac{\partial k^\varepsilon}{\partial x} - \int_0^1 \rho^\varepsilon \frac{\partial \rho^\varepsilon}{\partial x}\frac{\partial^2 k^\varepsilon}{\partial x^2} \\ & - \left(\alpha \int_0^1 \rho^\varepsilon + a^\varepsilon(t)\right) \int_0^1 \frac{\partial^2 k^\varepsilon}{\partial x^2}\frac{\partial \rho^\varepsilon}{\partial x}. \end{aligned}$$

Now, we proceed in the same way as for the previous equation, but we multiply the second equation of the system (4.17) by $\dfrac{\partial k^\varepsilon}{\partial x}$. For every $t \in (0,T)$, we obtain :

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\|\frac{\partial k^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} + \varepsilon \left\|\frac{\partial^2 k^\varepsilon}{\partial x^2}(t)\right\|^2_{L^2(\mathbb{T})} = & -\int_0^1 (\frac{\partial \rho^\varepsilon}{\partial x})^2 \frac{\partial k^\varepsilon}{\partial x} - \int_0^1 \rho^\varepsilon \frac{\partial k^\varepsilon}{\partial x}\frac{\partial^2 \rho^\varepsilon}{\partial x^2} \\ & - \left(\alpha \int_0^1 \rho^\varepsilon + a^\varepsilon(t)\right) \int_0^1 \frac{\partial^2 \rho^\varepsilon}{\partial x^2}\frac{\partial k^\varepsilon}{\partial x}. \end{aligned}$$

Adding the two previous equations, thanks to the periodicity of ρ^ε and $\dfrac{\partial k^\varepsilon}{\partial x}$, we infer that :

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\|\frac{\partial \rho^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} + \frac{1}{2}\frac{d}{dt}\left\|\frac{\partial k^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} \leq & -\int_0^1 (\frac{\partial \rho^\varepsilon}{\partial x})^2 \frac{\partial k^\varepsilon}{\partial x} - \int_0^1 \frac{\partial}{\partial x}\left(\rho^\varepsilon \frac{\partial \rho^\varepsilon}{\partial x}\frac{\partial k^\varepsilon}{\partial x}\right) \\ & - \left(\alpha \int_0^1 \rho^\varepsilon + a^\varepsilon(t)\right) \int_0^1 \frac{\partial}{\partial x}\left(\frac{\partial \rho^\varepsilon}{\partial x}\frac{\partial k^\varepsilon}{\partial x}\right) \\ & \leq -\int_0^1 (\frac{\partial \rho^\varepsilon}{\partial x})^2 \frac{\partial k^\varepsilon}{\partial x} \leq 0. \end{aligned}$$

We integrate in time and we use the fact that $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R} \times [0,T))$ and Lemma

2.3. We obtain in particular

$$\sup_{t\in(0,T)}\left\|\frac{\partial\rho^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})}+\sup_{t\in(0,T)}\left\|\frac{\partial k^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})}\leq C\left(\left\|\frac{\partial(\rho_0^+-\rho_0^-)}{\partial x}\right\|^2_{L^2(\mathbb{T})}+\left\|\frac{\partial(\rho_0^++\rho_0^-)}{\partial x}\right\|^2_{L^2(\mathbb{T})}\right).$$

That leads to the desired result. □

***Lemma* 4.2 (L^2 estimates of the solutions)** *Assume $(H1)$, $(H2)$, $(H3)$ and $\rho_0^\pm\in H^1_{loc}(\mathbb{R})$, if $\rho^{\pm,\varepsilon}\in C^\infty(\mathbb{R}\times[0,T))$ are solutions of the system (2.5)-(2.6) for every $T\geq 0$, then*

$$\left\|\rho^{+,\varepsilon}\right\|^2_{L^\infty((0,T);L^2(0,1))}+\left\|\rho^{-,\varepsilon}\right\|^2_{L^\infty((0,T);L^2(0,1))}\leq C\left(M_0+(B_0+\|a\|^2_{L^\infty(0,T)})\right)e^{4L_0(1+\alpha^2)T},$$

where B_0 is defined in Lemma 4.1, and $M_0=\left(\left\|\rho_0^+\right\|^2_{L^2(0,1)}+\left\|\rho_0^-\right\|^2_{L^2(0,1)}\right)$.

Proof of Lemma 4.2

We will use the same procedure of the proof of Lemma 4.1. We multiply the first equation of the system (4.17) by ρ^ε then we integrate in space. For every $t\in(0,T)$, we obtain :

$$\frac{1}{2}\frac{d}{dt}\|\rho^\varepsilon(t)\|^2_{L^2(\mathbb{T})}+\varepsilon\left\|\frac{\partial\rho^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})}=-\int_0^1(\rho^\varepsilon)^2\frac{\partial k^\varepsilon}{\partial x}-\left(\alpha\int_0^1\rho^\varepsilon+a^\varepsilon(t)\right)\int_0^1\rho^\varepsilon\frac{\partial k^\varepsilon}{\partial x}.$$

Similarly, we multiply the second equation of the system (4.17) by k^ε and we integrate in space. For every $t\in(0,T)$, we obtain :

$$\frac{1}{2}\frac{d}{dt}\|k^\varepsilon(t)\|^2_{L^2(0,1)}+\varepsilon\left\|\frac{\partial k^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})}=-\int_0^1\rho^\varepsilon\frac{\partial\rho^\varepsilon}{\partial x}k^\varepsilon-\left(\alpha\int_0^1\rho^\varepsilon+a^\varepsilon(t)\right)\int_0^1k^\varepsilon\frac{\partial\rho^\varepsilon}{\partial x}.$$

Now, we add the two previous equations and get :

$$\begin{aligned}\frac{1}{2}\left(\frac{d}{dt}\|\rho^\varepsilon(t)\|^2_{L^2(\mathbb{T})}+\frac{d}{dt}\|k^\varepsilon(t)\|^2_{L^2(0,1)}\right)&\leq-\int_0^1\left((\rho^\varepsilon)^2\frac{\partial k^\varepsilon}{\partial x}+\frac{1}{2}k^\varepsilon\frac{\partial(\rho^\varepsilon)^2}{\partial x}\right)\\&\quad-\left(\alpha\int_0^1\rho^\varepsilon+a^\varepsilon(t)\right)\left(\int_0^1k^\varepsilon\frac{\partial\rho^\varepsilon}{\partial x}+\int_0^1\rho^\varepsilon\frac{\partial k^\varepsilon}{\partial x}\right)\\&\leq-\frac{1}{2}\int_0^1(\rho^\varepsilon)^2\frac{\partial k^\varepsilon}{\partial x}-\frac{1}{2}\int_0^1\frac{\partial((\rho^\varepsilon)^2k^\varepsilon)}{\partial x}\\&\quad-\left(\alpha\int_0^1\rho^\varepsilon+a^\varepsilon(t)\right)\int_0^1\frac{\partial(k^\varepsilon\rho^\varepsilon)}{\partial x}.\end{aligned}$$

Recall that ρ^ε is periodic and k^ε is non-decreasing, we see that

$$\frac{1}{2}(\frac{d}{dt}\|\rho^\varepsilon(t)\|^2_{L^2(\mathbb{T})} + \frac{d}{dt}\|k^\varepsilon(t)\|^2_{L^2(0,1)}) \leq -\left(\alpha\int_0^1 \rho^\varepsilon + a^\varepsilon(t)\right)\int_0^1 \frac{\partial(k^\varepsilon\rho^\varepsilon)}{\partial x}.$$

But we know from $(H1)$ that ρ^ε and $(k^\varepsilon - 2L_0x)$ are 1-periodic functions, which implies that :

$$\int_0^1 \frac{\partial(k^\varepsilon\rho^\varepsilon)}{\partial x} = \int_0^1 \frac{\partial((k^\varepsilon - 2L_0x)\rho^\varepsilon)}{\partial x} + 2L_0\int_0^1 \frac{\partial(x\rho^\varepsilon)}{\partial x} = 2L_0\int_0^1 x\frac{\partial\rho^\varepsilon}{\partial x} + 2L_0\int_0^1 \rho^\varepsilon.$$

We use Lemmas 4.1, 2.3, and the fact that ($ab \leq \frac{1}{2}(a^2+b^2)$ and $(a+b)^2 \leq 2(a^2+b^2)$), to deduce that :

$$\begin{aligned}
\frac{d}{dt}\left(\|k^\varepsilon(t)\|^2_{L^2(0,1)} + \|\rho^\varepsilon(t)\|^2_{L^2(\mathbb{T})}\right) &\leq 4L_0\left(|\alpha|\,\|\rho^\varepsilon(t)\|_{L^2(\mathbb{T})} + \|a\|_{L^\infty(0,T)}\right)\left(\left\|\frac{\partial\rho^\varepsilon}{\partial x}(t)\right\|_{L^2(\mathbb{T})} + \|\rho^\varepsilon(t)\|_{L^2(\mathbb{T})}\right)\\
&\leq 4L_0\left(\|a\|^2_{L^\infty(0,T)} + (1+\alpha^2)\|\rho^\varepsilon(t)\|^2_{L^2(\mathbb{T})} + \left\|\frac{\partial\rho^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})}\right)\\
&\leq 4L_0\left(CB_0 + \|a\|^2_{L^\infty(0,T)}\right)\\
&\quad +4L_0(1+\alpha^2)\left(\|\rho^\varepsilon(t)\|^2_{L^2(\mathbb{T})} + \|k^\varepsilon(t)\|^2_{L^2(0,1)}\right),
\end{aligned}$$

Using the previous estimate and the fact that $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}\times[0,T))$, we finally obtain :

$$\|\rho^\varepsilon\|^2_{L^\infty((0,T);L^2(\mathbb{T}))} + \|k^\varepsilon\|^2_{L^\infty((0,T)L^2(0,1))} \leq C\left(M_0 + B_0 + \|a\|^2_{L^\infty(0,T)}\right)e^{4L_0(1+\alpha^2)T}.$$

This leads to the desired result. □

***Lemma* 4.3 (L^2 estimate on the time derivatives of the solutions)** *Assume $(H1)$, $(H2)$, $(H3)$ and $\rho_0^\pm \in H^1_{loc}(\mathbb{R})$, if $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}\times[0,T))$ is a solution of the system (2.5)-(2.6) for every $T \geq 0$, then there exists a constant $C\left(T, L_0, \alpha, \|a\|_{L^\infty(0,T)}, M_0, B_0\right)$ independent of ε such that :*

$$\left\|\frac{\partial\rho^{\pm,\varepsilon}}{\partial t}\right\|_{L^2(\mathbb{T}\times(0,T))} \leq C.$$

Proof of Lemma 4.3
For the proof of Lemma 4.3, it is sufficient to show that the second term of the system (2.5)

$$f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon(t)] = \mp\left(a^\varepsilon(t) + \rho^\varepsilon + \alpha\int_0^1 \rho^\varepsilon dx\right)\frac{\partial\rho^{\pm,\varepsilon}}{\partial x},$$

is bounded in $L^\infty((0,T);L^2(\mathbb{T}))$ uniformly in ε. Indeed,

$$\begin{aligned}\left\|f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon]\right\|_{L^\infty((0,T^\star);L^2(\mathbb{T}))} &\leq \left\|\left(a^\varepsilon(\cdot) + \rho^\varepsilon + \alpha\int_0^1 \rho^\varepsilon dx\right)\frac{\partial\rho^{\pm,\varepsilon}}{\partial x}\right\|_{L^\infty((0,T);L^2(\mathbb{T}))}\\ &\leq C\left(\|\rho^\varepsilon\|_{L^\infty(\mathbb{T}\times(0,T))} + \|a\|_{L^\infty(0,T)}\right)\left\|\frac{\partial\rho^{\pm,\varepsilon}}{\partial x}\right\|_{L^\infty((0,T);L^2(\mathbb{T}))}.\end{aligned}$$

We use the Lemmas 4.1, 4.2 and the Sobolev injections to deduce that there exists a constant $C\left(T, L_0, \alpha, \|a\|_{L^\infty(0,T)}, M_0, B_0\right)$ such that

$$\left\|f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon]\right\|_{L^\infty((0,T);L^2(\mathbb{T}))} \leq C.$$

To end up, we multiply the first and the second equations of the system (2.5) by $\dfrac{\partial\rho^{+,\varepsilon}}{\partial t}$, $\dfrac{\partial\rho^{-,\varepsilon}}{\partial t}$ respectively and we integrate in space. We deduce that for every $t \in (0,T)$ we have :

$$\left\|\frac{\partial\rho^{\pm,\varepsilon}}{\partial t}(t)\right\|^2_{L^2(\mathbb{T})} + \frac{\varepsilon}{2}\frac{d}{dt}\left\|\frac{\partial\rho^{\pm,\varepsilon}}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} = \int_0^1 f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon(t)]\frac{\partial\rho^{\pm,\varepsilon}}{\partial t}.$$

We integrate in time and we use the fact that $\rho^{\pm,\varepsilon} \in C(\mathbb{R}\times[0,T))$ for all $T \geq 0$, we get :

$$\left\|\frac{\partial\rho^{\pm,\varepsilon}}{\partial t}\right\|^2_{L^2(\mathbb{T}\times(0,T))} + \frac{\varepsilon}{2}\left\|\frac{\partial\rho^{\pm,\varepsilon}}{\partial x}(T)\right\|^2_{L^2(\mathbb{T})} = \int_0^T\int_0^1 f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon(t)]\frac{\partial\rho^{\pm,\varepsilon}}{\partial t} + \frac{\varepsilon}{2}\left\|\frac{\partial\rho_0{}^{\pm,\varepsilon}}{\partial x}\right\|^2_{L^2(\mathbb{T})}.$$

We apply Hölder's inequality and the fact that $\varepsilon < 1$ and $ab \leq \frac{1}{2}(a^2+b^2)$, to obtain that :

$$\begin{aligned}\left\|\frac{\partial\rho^{\pm,\varepsilon}}{\partial t}\right\|^2_{L^2(\mathbb{T}\times(0,T))} &\leq \left\|f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon]\right\|_{L^2(\mathbb{T}\times(0,T))}\left\|\frac{\partial\rho^{\pm,\varepsilon}}{\partial t}\right\|_{L^2(\mathbb{T}\times(0,T))} + \frac{1}{2}\left\|\frac{\partial\rho_0{}^{\pm,\varepsilon}}{\partial x}\right\|^2_{L^2(\mathbb{T})}\\ &\leq \frac{C}{2}\left(\left\|f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon]\right\|^2_{L^2(\mathbb{T}\times(0,T))} + \left\|\frac{\partial\rho^{\pm,\varepsilon}}{\partial t}\right\|^2_{L^2(\mathbb{T}\times(0,T))} + \left\|\frac{\partial\rho_0{}^{\pm}}{\partial x}\right\|^2_{L^2(\mathbb{T})}\right).\end{aligned}$$

that leads to

$$\begin{aligned}\left\|\frac{\partial\rho^{\pm,\varepsilon}}{\partial t}\right\|^2_{L^2(\mathbb{T}\times(0,T))} &\leq C\left(\left\|f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon]\right\|^2_{L^2(\mathbb{T}\times(0,T))}+\left\|\frac{\partial{\rho_0}^{\pm}}{\partial x}\right\|^2_{L^2(\mathbb{T})}\right)\\ &\leq C\left(T\left\|f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon]\right\|^2_{L^\infty((0,T);L^2(\mathbb{T}))}+\left\|\frac{\partial{\rho_0}^{\pm}}{\partial x}\right\|^2_{L^2(\mathbb{T})}\right)\leq C,\end{aligned}$$

where $C\left(T, L_0, \alpha, \|a\|_{L^\infty(0,T)}, M_0, B_0\right)$. □

Remark 4.4 (The sense of the initial conditions) *According to Lemma 4.3, we have $\rho^{\pm,\varepsilon,per}\in C([0,T),L^2(\mathbb{T}))$ uniformly in ε. This will give a sense to the limit of the initial conditions.*

Theorem 4.5 (Long time existence) *Assume $(H1)$, $(H2)$ and $(H3)$, for all L_0, $T\geq 0$, $\alpha\in\mathbb{R}$ and $\rho_0^{\pm}\in H^1_{loc}(\mathbb{R})$, the system (2.5)-(2.6) admits the solutions $\rho^{\pm,\varepsilon}\in C^\infty(\mathbb{R}\times[0,T))$, with $\rho^{\pm,\varepsilon}(.,t)$ verifying $(H1)$ and $(H2)$. Moreover, there exists a constant $C\left(T, L_0, \alpha, \|a\|_{L^\infty(0,T)}, M_0, B_0\right)$ independent of ε, with B_0 and M_0 defined in Lemmas 4.1 and 4.2 respectively, such that :*

$$\left\|\rho^{\pm,\varepsilon,per}\right\|_{L^\infty((0,T);L^2(\mathbb{T}))}+\left\|\frac{\partial\rho^{\pm,\varepsilon}}{\partial x}\right\|_{L^\infty((0,T);L^2(\mathbb{T}))}+\left\|\frac{\partial\rho^{\pm,\varepsilon}}{\partial t}\right\|_{L^2(\mathbb{T}\times(0,T))}\leq C, \tag{4.18}$$

where $\rho^{\pm,\varepsilon,per}=\rho^{\pm,\varepsilon}-L_0x$.

Proof of Theorem 4.5
We are going to prove that local time solutions obtained by Corollary 3.3 can be extended to global time solutions for the same system.
We argue by contradiction : Assume that there exists a maximum time T_{max} such that, we have the existence of solutions of the system (2.5)-(2.6) in the function space $C^\infty(\mathbb{R}\times[0,T_{max}))$.
For every $\delta>0$, we consider the system (2.5) with the initial conditions

$$\rho^{\pm,\varepsilon}_{\delta,max}=\rho^{\pm,\varepsilon}(x,T_{max}-\delta).$$

We apply for the second time the same technic of Corollary 3.3 to deduce that there exists a time

$$T^\star_{\delta,max}(\|\rho^{\pm,\varepsilon,per}_{\delta,max}\|_{H^1(\mathbb{T})},\|a\|_{L^\infty(0,T)},L_0,\alpha,\varepsilon)>0,\ \text{where}\ \rho^{\pm,\varepsilon,per}_{\delta,max}=\rho^{\pm,\varepsilon}_{\delta,max}-L_0x,$$

such that the system (2.5)-(2.6) admits a solution defined until the time

$$T_0=(T_{max}-\delta)+T^\star_{\delta,max}.$$

Moreover, according to Lemmas 4.1 and 4.2, we know that $\rho^{\pm,\varepsilon,per}_{\delta,max}$ are δ-uniformly bounded in $H^1(\mathbb{T})$. We use (2.16) to deduce that there exists a constant

$$C(\varepsilon, T_{max}, \alpha, \|a\|_{L^\infty(0,T)}, L_0) > 0,$$

independent of δ such that $T^\star_{\delta,max} \geq C > 0$, then $\lim_{\delta\to 0} T^\star_{\delta,max} \geq C > 0$ which implies that $T_0 > T_{max}$ and so a contradiction.
The estimation (4.18) is a consequence of Lemmas 4.1, 4.2 and 4.3. □

5 Existence and uniqueness of the solution of (1.1)-(1.2)

In this paragraph, we are going to prove that the system (1.1)-(1.2) admits a unique solution $\rho^\pm$ (in the distribution sense) which is the limit as $\varepsilon \to 0$ of $\rho^{\pm,\varepsilon}$ given by Theorem 4.5. In order to do that, we pass to the limit when ε tends to 0 in the system (2.7)-(2.8), and we use (4.18) in order to assure the compactness. The proof of the uniqueness uses direct arguments.
Proof of Theorem 1.1
We first prove the existence and then establish the uniqueness.
Step 1 (Existence) :

Let $\rho^{\pm,\varepsilon}$ be the solution of the system (2.5) given by Theorem 4.5. According to (4.18) we know that $\rho^{\pm,\varepsilon,per}$ are ε-uniformly bounded in $H^1(\mathbb{T}\times(0,T))$, then we can extract a sub-sequence that converges weakly in $H^1(\mathbb{T}\times(0,T))$. Knowing that $H^1(\mathbb{T}\times(0,T))$ is compact in $L^2(\mathbb{T}\times(0,T))$, this sub-sequence strongly converges in $L^2(\mathbb{T}\times(0,T))$. If we denote by $\rho^{\pm,per}$ the limit of this sub-sequence, we have to prove that $\rho^{\pm,per} + L_0 x$ is a solution of the system (1.1)-(1.2) in the sense of distribution. Indeed, by Lemma 2.3, the term $\mp(L_0 a^\varepsilon)$ of the equation (2.7) converges strongly to $(\mp L_0 a)$ in $L^2(0,T)$.
The linear term

$$\mp\left(L_0 C_\alpha[\rho^\varepsilon] + a^\varepsilon(t)\frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x}\right)$$

of the equation (2.7), weakly converges in $L^1(\mathbb{T}\times(0,T))$ and the reason is, in the one hand that $\frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x}$ are ε-uniformly bounded in $L^2(\mathbb{T}\times(0,T))$ that gives us the weak convergence in $L^2(\mathbb{T}\times(0,T))$ and on the other hand, that a^ε strongly converges in $L^2(0,T)$. Then, the linear term converges in the sense of distributions (i.e. in $\mathcal{D}'(\mathbb{T}\times(0,T))$). It remains to prove that the bilinear term

$$C_\alpha[\rho^\varepsilon]\frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x}$$

of the equation (2.7), also converges in the sense of distributions. We have :

1. The sequence $C_\alpha[\rho^\varepsilon]$ is compact in $L^2(\mathbb{T}\times(0,T))$.
2. The functions $\dfrac{\partial \rho^{\pm,\varepsilon,per}}{\partial x}$ are ε-uniformly bounded in $L^2(\mathbb{T}\times(0,T))$,

that gives us a strong convergence in $L^2(\mathbb{T}\times(0,T))$ times a weak convergence in $L^2(\mathbb{T}\times(0,T))$ and hence a weak convergence of the product in $L^1(\mathbb{T}\times(0,T))$. This leads, as a consequence, to the convergence in the distribution sense. This, altogether, shows that $\rho^{\pm,per}+L_0x$ is a solution in the sense of distribution of the system (1.1)-(1.2) and $\rho^{\pm,per}$ verifies estimate (4.18).

It remains to prove that the initial condition is satisfied by the limit function $\rho^{\pm,per}$. In fact, according to the estimate (4.18) on $\rho^{\pm,\varepsilon,per}$, $\dfrac{\partial \rho^{\pm,\varepsilon}}{\partial t}$ and $\dfrac{\partial \rho^{\pm,\varepsilon}}{\partial x}$, we see that $\rho^{\pm,\varepsilon,per}$ is ε-uniformly bounded in $H^1(\mathbb{T}\times(0,T))$.
From the fact that the injection of $H^1(\mathbb{T}\times(0,T))$ in $C([0,T);L^2(\mathbb{T}))$ is continuous and compact by classical arguments, we see that, for all $v\in L^2(\mathbb{T})$, the application $\gamma : U \longmapsto \displaystyle\int_0^1 U(0)v$ is a continuous linear form for $U\in C([0,T);L^2(\mathbb{T}))$ and hence $\gamma(\rho^{\pm,\varepsilon,per})\to\gamma(\rho^{\pm,per})$ as $\varepsilon\to 0$, because up to a subsequence $\rho^{\pm,\varepsilon,per}$ converges strongly in $C([0,T);L^2(\mathbb{T}))$. This altogether proves that the solution verifies the initial conditions (1.2).
Step 2 (Uniqueness) :

Let $\rho_1^\pm$ and $\rho_2^\pm$ be two solutions of the system (1.1), such that $\rho_1^\pm(\cdot,0)=\rho_2^\pm(\cdot,0)=\rho_0^\pm$ and $\rho_i^\pm(\cdot,t)$ verify $(H1)$, $(H2)$ and estimate (4.18) for $i=1,2$, $t\in(0,T)$.

If we denote $\rho_i=\rho_i^+-\rho_i^-$, $k_i=\rho_i^++\rho_i^-$ for $i=1,2$, then it is clear that $(\rho_1-\rho_2)$ and (k_1-k_2) are 1-periodic functions in space and ρ_i, k_i verify the following system for $i=1,2$:

$$\begin{cases} \dfrac{\partial \rho_i}{\partial t} &= -\left(\rho_i+\alpha\displaystyle\int_0^1\rho_i dx+a(t)\right)\dfrac{\partial k_i}{\partial x} \quad \text{in} \quad \mathcal{D}'(\mathbb{R}\times(0,T)),\\ \dfrac{\partial k_i}{\partial t} &= -\left(\rho_i+\alpha\displaystyle\int_0^1\rho_i dx+a(t)\right)\dfrac{\partial \rho_i}{\partial x} \quad \text{in} \quad \mathcal{D}'(\mathbb{R}\times(0,T)). \end{cases} \tag{5.19}$$

We subs-tract the two systems to obtain that :

$$\begin{cases} \dfrac{\partial(\rho_1-\rho_2)}{\partial t} &= -\left(\rho_1+\alpha\displaystyle\int_0^1\rho_1 dx\right)\dfrac{\partial k_1}{\partial x}+\left(\rho_2+\alpha\displaystyle\int_0^1\rho_2 dx\right)\dfrac{\partial k_2}{\partial x}-a(t)\dfrac{\partial(k_1-k_2)}{\partial x},\\ \dfrac{\partial(k_1-k_2)}{\partial t} &= -\left(\rho_1+\alpha\displaystyle\int_0^1\rho_1 dx\right)\dfrac{\partial \rho_1}{\partial x}+\left(\rho_2+\alpha\displaystyle\int_0^1\rho_2 dx\right)\dfrac{\partial \rho_2}{\partial x}-a(t)\dfrac{\partial(\rho_1-\rho_2)}{\partial x}. \end{cases}$$

The previous system is equivalent to :

$$\begin{cases} \dfrac{\partial(\rho_1-\rho_2)}{\partial t} &= -\left((\rho_1-\rho_2)+\alpha\displaystyle\int_0^1(\rho_1-\rho_2)dx\right)\dfrac{\partial k_1}{\partial x}-\left(\rho_2+\alpha\displaystyle\int_0^1\rho_2 dx\right)\dfrac{\partial(k_1-k_2)}{\partial x}\\ &\quad -a(t)\dfrac{\partial(k_1-k_2)}{\partial x},\\ \dfrac{\partial(k_1-k_2)}{\partial t} &= -\left((\rho_1-\rho_2)+\alpha\displaystyle\int_0^1(\rho_1-\rho_2)dx\right)\dfrac{\partial \rho_1}{\partial x}-\left(\rho_2+\alpha\displaystyle\int_0^1\rho_2 dx\right)\dfrac{\partial(\rho_1-\rho_2)}{\partial x}\\ &\quad -a(t)\dfrac{\partial(\rho_1-\rho_2)}{\partial x}. \end{cases}$$

We multiply the first equation of this system by $(\rho_1-\rho_2)$ and we integrate in space to obtain, for almost every t, that :

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|(\rho_1-\rho_2)(t)\|^2_{L^2(\mathbb{T})} = & -\int_0^1\left((\rho_1-\rho_2)^2\frac{\partial k_1}{\partial x}\right)-\alpha\left(\int_0^1(\rho_1-\rho_2)\right)\int_0^1\left((\rho_1-\rho_2)\frac{\partial k_1}{\partial x}\right)\\ & -\int_0^1\left((\rho_1-\rho_2)\left(\rho_2+\alpha\int_0^1\rho_2\right)\frac{\partial(k_1-k_2)}{\partial x}\right)\\ & -a(t)\int_0^1\left((\rho_1-\rho_2)\frac{\partial(k_1-k_2)}{\partial x}\right). \end{aligned}$$

Similarly, we multiply the second equation by (k_1-k_2) and we integrate in space to get for almost every time t :

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|(k_1-k_2)(t)\|^2_{L^2(\mathbb{T})} = & -\int_0^1\left((\rho_1-\rho_2)(k_1-k_2)\frac{\partial \rho_1}{\partial x}\right)-\alpha\left(\int_0^1(\rho_1-\rho_2)\right)\int_0^1(k_1-k_2)\frac{\partial \rho_1}{\partial x}\\ & -\int_0^1\left((k_1-k_2)\left(\rho_2+\alpha\int_0^1\rho_2\right)\frac{\partial(\rho_1-\rho_2)}{\partial x}\right)\\ & -a(t)\int_0^1\left((k_1-k_2)\frac{\partial(\rho_1-\rho_2)}{\partial x}\right). \end{aligned}$$

We add the two previous equations to obtain, for almost every time t :

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|(\rho_1-\rho_2)(t)\|^2_{L^2(\mathbb{T})}+\|(k_1-k_2)(t)\|^2_{L^2(\mathbb{T})}\right)\\
=\ & -\int_0^1\left((\rho_1-\rho_2)^2\frac{\partial k_1}{\partial x}\right)-\alpha\left(\int_0^1(\rho_1-\rho_2)\right)\int_0^1\left((\rho_1-\rho_2)\frac{\partial k_1}{\partial x}\right)\\
& -\alpha\left(\int_0^1(\rho_1-\rho_2)\right)\int_0^1\left((k_1-k_2)\frac{\partial\rho_1}{\partial x}\right)-\int_0^1\left(\frac{\partial}{\partial x}\left((\rho_1-\rho_2)(k_1-k_2)\left(\rho_2+\alpha\int_0^1\rho_2\right)\right)\right)\\
& -\int_0^1\left((\rho_1-\rho_2)(k_1-k_2)\frac{\partial(\rho_1-\rho_2)}{\partial x}\right)-a(t)\int_0^1\left(\frac{\partial}{\partial x}\left((\rho_1-\rho_2)(k_1-k_2)\right)\right).
\end{aligned}$$

From the fact that ρ_i, $i=1,2$ and (k_1-k_2) are 1-periodic functions in space, the previous equation becomes :

$$\begin{aligned}
=\ & \overbrace{-\int_0^1\left((\rho_1-\rho_2)^2\frac{\partial k_1}{\partial x}\right)}^{I_1}\ \overbrace{-\alpha\left(\int_0^1(\rho_1-\rho_2)\right)\int_0^1\left((\rho_1-\rho_2)\frac{\partial k_1}{\partial x}\right)}^{I_2}\\
& \overbrace{-\alpha\left(\int_0^1(\rho_1-\rho_2)\right)\int_0^1\left((k_1-k_2)\frac{\partial\rho_1}{\partial x}\right)}^{I_3}\ \overbrace{-\int_0^1\left((\rho_1-\rho_2)(k_1-k_2)\frac{\partial(\rho_1-\rho_2)}{\partial x}\right)}^{I_4}.
\end{aligned}$$

And since $\dfrac{\partial k_i}{\partial x}\geq 0$ for $i=1,2$, we know that :

$$\begin{aligned}
I_1+I_4 &= -\int_0^1\left((\rho_1-\rho_2)^2\frac{\partial k_1}{\partial x}\right)-\frac{1}{2}\int_0^1\left((k_1-k_2)\frac{\partial}{\partial x}\left((\rho_1-\rho_2)^2\right)\right)\\
&= -\int_0^1\left((\rho_1-\rho_2)^2\frac{\partial k_1}{\partial x}\right)+\frac{1}{2}\int_0^1\left((\rho_1-\rho_2)^2\frac{\partial(k_1-k_2)}{\partial x}\right)\\
&= -\frac{1}{2}\int_0^1\left((\rho_1-\rho_2)^2\frac{\partial(k_1+k_2)}{\partial x}\right)\leq 0.
\end{aligned}$$

Moreover, from (4.18), we have for almost every t :

$$\begin{aligned}
I_2 &\leq |\alpha|\|(\rho_1-\rho_2)(t)\|_{L^2(\mathbb{T})}\|(\rho_1-\rho_2)(t)\|_{L^2(\mathbb{T})}\left\|\frac{\partial k_1}{\partial x}(t)\right\|_{L^2(\mathbb{T})}\\
&\leq C\|(\rho_1-\rho_2)(t)\|^2_{L^2(\mathbb{T})}.
\end{aligned}$$

Similarly, from (4.18), we have, for almost every t, that :

$$\begin{aligned} I_3 &\leq |\alpha| \|(\rho_1 - \rho_2)(t)\|_{L^2(\mathbb{T})} \|(k_1 - k_2)(t)\|_{L^2(\mathbb{T})} \left\| \frac{\partial \rho_1}{\partial x}(t) \right\|_{L^2(\mathbb{T})} \\ &\leq C \left(\|(\rho_1 - \rho_2)(t)\|^2_{L^2(\mathbb{T})} + \|(k_1 - k_2)(t)\|^2_{L^2(\mathbb{T})} \right) \end{aligned}$$

Then

$$\frac{d}{dt} \left(\|(\rho_1 - \rho_2)(t)\|^2_{L^2(\mathbb{T})} + \|(k_1 - k_2)(t)\|^2_{L^2(\mathbb{T})} \right) \leq C \left(\|(\rho_1 - \rho_2)(t)\|^2_{L^2(\mathbb{T})} + \|(k_1 - k_2)(t)\|^2_{L^2(\mathbb{T})} \right).$$

Now, we integrate in time and we use the fact that ρ_i, $k_i \in C([0,T), L^2_{loc}(\mathbb{R}))$, $\rho_1(\cdot, 0) = \rho_2(\cdot, 0)$ and $k_1(\cdot, 0) = k_2(\cdot, 0)$ to obtain that :

$$\sup_{t \in (0,T)} \|(\rho_1 - \rho_2)(t)\|^2_{L^2(\mathbb{T})} + \sup_{t \in (0,T)} \|(k_1 - k_2)(t)\|^2_{L^2(\mathbb{T})} \leq 0.$$

This achieves the proof of uniqueness. □

Remark 5.1 *In Theorem 1.1, we have proved a result of existence and uniqueness in $H^1_{loc}(\mathbb{R} \times [0,T))$ depending on some uniform estimates in this space. These estimates give a sufficient compactness in order to ensure the passage to the limit as ε tends to 0 in the bilinear term. However, the space $W^{1,1}_{loc}(\mathbb{R} \times [0,T))$ does not give enough compactness. On the other hand, the space of functions $L^2_{loc}(\mathbb{R} \times [0,T))$ having their derivatives in $L^\infty((0,T); (L^1 \log L^1)_{loc}(\mathbb{R}))$ requires the minimal properties to ensure the passage to the limit in the bilinear term. The result of existence in this space will be the core of a paper in preparation.*

6 Further properties : comparison principle with case $\alpha = 0$

In this section, we are going to prove a comparison principle result of the system (1.1) in the case $\alpha = 0$ (i.e. the Theorem 1.2). In order to do this, first we prove in the following subsection the same result for the approximate system (2.5). Then, we give the proof of Theorem 1.2.

6.1 Comparison principle for the regularized system with case $\alpha = 0$

***Lemma* 6.1 (Comparison principle)** *Let $a(\cdot)$ satisfies $(H3)$ and $\rho_1^{\pm,\varepsilon}$, $\rho_2^{\pm,\varepsilon} \in C^\infty(\mathbb{R} \times [0,T))$ be two solutions of the system (2.5) with $\alpha = 0$. Moreover, let $\rho_1^{\pm,\varepsilon}(.,t)$, $\rho_2^{\pm,\varepsilon}(.,t)$ verify $(H1)$ and $(H2)$ for all $t \in [0,T)$. Then, if $\rho_1^{\pm,\varepsilon}(\cdot,0) \leq \rho_2^{\pm,\varepsilon}(\cdot,0)$ in $\mathbb{R}$, we have $\rho_1^{\pm,\varepsilon} \leq \rho_2^{\pm,\varepsilon}$ on $\mathbb{R} \times [0,T)$.*

Proof of Lemma 6.1
We know that $\rho_1^{\pm,\varepsilon}$ and $\rho_2^{\pm,\varepsilon}$ verify the following systems :

$$\begin{cases} \dfrac{\partial \rho_1^{+,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 \rho_1^{+,\varepsilon}}{\partial x^2} = & -\left(\rho_1^{+,\varepsilon} - \rho_1^{-,\varepsilon} + a^\varepsilon(t)\right) \dfrac{\partial \rho_1^{+,\varepsilon}}{\partial x} \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0,T)), \\ \dfrac{\partial \rho_1^{-,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 \rho_1^{-,\varepsilon}}{\partial x^2} = & \left(\rho_1^{+,\varepsilon} - \rho_1^{-,\varepsilon} + a^\varepsilon(t)\right) \dfrac{\partial \rho_1^{-,\varepsilon}}{\partial x} \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0,T)), \end{cases}$$

$$\begin{cases} \dfrac{\partial \rho_2^{+,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 \rho_2^{+,\varepsilon}}{\partial x^2} = & -\left(\rho_2^{+,\varepsilon} - \rho_2^{-,\varepsilon} + a^\varepsilon(t)\right) \dfrac{\partial \rho_2^{+,\varepsilon}}{\partial x} \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0,T)), \\ \dfrac{\partial \rho_2^{-,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 \rho_2^{-,\varepsilon}}{\partial x^2} = & \left(\rho_2^{+,\varepsilon} - \rho_2^{-,\varepsilon} + a^\varepsilon(t)\right) \dfrac{\partial \rho_2^{-,\varepsilon}}{\partial x} \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0,T)), \end{cases}$$

respectively.

If we denote $w^{\pm,\varepsilon}$ by $\tilde{\rho}_2^{\pm,\varepsilon} - \tilde{\rho}_1^{\pm,\varepsilon}$, where,

$$\tilde{\rho}_2^{\pm,\varepsilon} = \rho_2^{\pm,\varepsilon} e^{-\gamma t}, \quad \text{and} \quad \tilde{\rho}_1^{\pm,\varepsilon} = \rho_1^{\pm,\varepsilon} e^{-\gamma t} \quad \text{with} \quad \gamma > 0,$$

we can easily check that $w^{\pm,\varepsilon}$ are solutions of the following system :

$$\begin{cases} \dfrac{\partial w^{+,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 w^{+,\varepsilon}}{\partial x^2} + \gamma w^{+,\varepsilon} = & -e^{\gamma t}\left(w^{+,\varepsilon} - w^{-,\varepsilon}\right) \dfrac{\partial \tilde{\rho}_2^{+,\varepsilon}}{\partial x} - e^{\gamma t}\left(\tilde{\rho}_1^{+,\varepsilon} - \tilde{\rho}_1^{-,\varepsilon} + e^{-\gamma t} a^\varepsilon(t)\right) \dfrac{\partial w^{+,\varepsilon}}{\partial x}, \\ \dfrac{\partial w^{-,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 w^{-,\varepsilon}}{\partial x^2} + \gamma w^{-,\varepsilon} = & e^{\gamma t}\left(w^{+,\varepsilon} - w^{-,\varepsilon}\right) \dfrac{\partial \tilde{\rho}_2^{-,\varepsilon}}{\partial x} + e^{\gamma t}\left(\tilde{\rho}_1^{+,\varepsilon} - \tilde{\rho}_1^{-,\varepsilon} + e^{-\gamma t} a^\varepsilon(t)\right) \dfrac{\partial w^{-,\varepsilon}}{\partial x}. \end{cases} \tag{6.20}$$

We are interested in the $\min\limits_{(k,x,t)\in\{+,-\}\times\mathbb{T}\times(0,T)} \left(w^{k,\varepsilon}(x,t)\right)$. Our result follows if we can prove that this minimum is positive. However, this minimum is attained at a point $(k_0, x_0, t_0) \in \{+,-\} \times \mathbb{T} \times [0,T]$ (because $w^{+,\varepsilon}$ et $w^{-,\varepsilon}$ are $C^\infty(\mathbb{T} \times (0,T))$).

Two cases may occur :

1. Case $t_0 = 0$. We have

$$\min_{(k,x,t)\in\{+,-\}\times\mathbb{T}\times(0,T)} (w^{k,\varepsilon}(x,t)) = w^{k_0,\varepsilon}(x_0,t_0) = \left(\rho_2^{k_0,\varepsilon}(x_0,0) - \rho_1^{k_0,\varepsilon}(x_0,0)\right) e^{-\gamma t_0} \geq 0;$$

and we are done.

2. Case $t_0 \in (0, T]$. We have : (k_0, x_0, t_0) is a minimum point, then :

$$\frac{\partial^2 w^{k_0,\varepsilon}}{\partial x^2}(x_0, t_0) \geq 0, \tag{6.21}$$

$$\frac{\partial w^{k_0,\varepsilon}}{\partial t}(x_0, t_0) \leq 0, \tag{6.22}$$

$$\frac{\partial w^{k_0,\varepsilon}}{\partial x}(x_0, t_0) = 0. \tag{6.23}$$

We combine (6.21), (6.22), (6.23) and we take into consideration that $w^{\pm,\varepsilon}$ verifies the system (6.20), we obtain that :

$$\begin{aligned}\gamma w^{k_0,\varepsilon}(x_0, t_0) &\geq e^{\gamma t_0} sign(w^{+,\varepsilon}(x_0, t_0) - w^{-,\varepsilon}(x_0, t_0))(w^{+,\varepsilon}(x_0, t_0) - w^{-,\varepsilon}(x_0, t_0))\frac{\partial \tilde{\rho}_2^{k_0,\varepsilon}}{\partial x} \\ &\geq e^{\gamma t_0}|w^{+,\varepsilon}(x_0, t_0) - w^{-,\varepsilon}(x_0, t_0)|\frac{\partial \tilde{\rho}_2^{k_0,\varepsilon}}{\partial x} \geq 0.\end{aligned}$$

Then $\tilde{\rho}_1^{\pm,\varepsilon} \leq \tilde{\rho}_2^{\pm,\varepsilon}$ in $\mathbb{R} \times (0, T)$, which gives $\rho_1^{\pm,\varepsilon} \leq \rho_2^{\pm,\varepsilon}$. □

We now give the proof of Theorem 1.2 :

6.2 Proof of Theorem 1.2

Let

$$\rho_1^{\pm}(x, 0) = \rho_{1,0}^{\pm}(x) = \rho_{1,0}^{\pm,per}(x) + L_0 x \quad \text{and} \quad \rho_2^{\pm}(x, 0) = \rho_{2,0}^{\pm}(x) = \rho_{2,0}^{\pm,per}(x) + L_0 x.$$

If we denote

$$\rho_{1,0}^{\pm,\varepsilon}(x) = \rho_{1,0}^{\pm,per} * \eta_\varepsilon(x) + L_0 x \quad \text{and} \quad \rho_{2,0}^{\pm,\varepsilon}(x) = \rho_{2,0}^{\pm,per} * \eta_\varepsilon(x) + L_0 x,$$

where η_ε is a regularization sequence, we can easily check that $\rho_{1,0}^{\pm,\varepsilon} \leq \rho_{2,0}^{\pm,\varepsilon}$. Moreover, according to the uniqueness of the solution, we know that there exist $\rho_1^{\pm,\varepsilon}$, $\rho_2^{\pm,\varepsilon} \in C^\infty(\mathbb{R} \times [0, T))$, verifying $(H2)$ for all $t \in (0, T)$, that are solutions of the system (2.5), such that

$$\rho_1^{\pm} = \lim_{\varepsilon \to 0} \rho_1^{\pm,\varepsilon}, \quad \rho_2^{\pm} = \lim_{\varepsilon \to 0} \rho_2^{\pm,\varepsilon},$$

$$\rho_1^{\pm,\varepsilon}(x, 0) = \rho_{1,0}^{\pm,\varepsilon}(x) \quad \text{and} \quad \rho_2^{\pm,\varepsilon}(x, 0) = \rho_{2,0}^{\pm,\varepsilon}(x).$$

We apply Lemma 6.1 to obtain that $\rho_1^{\pm,\varepsilon} \leq \rho_2^{\pm,\varepsilon}$. We pass to the limit as $\varepsilon \to 0$ to deduce that $\rho_1^{\pm} \leq \rho_2^{\pm}$ a.e. in $\mathbb{R} \times (0, T)$. □

Remark 6.2 *Thanks to this comparison result, we proved in a previous paper [48] the existence and the uniqueness of a solution (in the viscosity sense). Here, this comparison result is an indirect explanation of our estimates obtained in Lemmas 4.1, 4.2 and 4.3 that have ensured our principal Theorem 1.1.*

7 Application in the case of classical Burgers equation

In this paragraph we are going to prove that this technic can be also applied to the classical Burgers equation, even in the frame of functions in $W^{1,p}_{loc}(\mathbb{R}\times(0,T))$ for all $1 \leq p < +\infty$, constituting the proof of Theorem 1.4 :

Proof of Theorem 1.4

First, we remark that the existence of solution to the regularized problem con br done thanks to the continuous injection $W^{1,p}(\mathbb{R})$ in $L^\infty(\mathbb{R})$.

Now, for the proof of this theorem, it suffices to show an estimation over the space derivatives of the solution (i.e. a result similar to that of Lemma 4.1).

First of all, we put ourselves in the hypothesis of Lemma 4.1. We derive the equation (1.4) with respect to x, then we multiply it by $\left(\frac{\partial u}{\partial x}\right)^{p-1}$ and finally we integrate over $\mathbb{R}$, since u verifies $(H2)$, we obtain that :

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\|\frac{\partial u}{\partial x}(t)\|^p_{L^p(\mathbb{R})} &= -\int_{\mathbb{R}} f''(u)\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial x}\right)^p - \int_{\mathbb{R}} f'(u)\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial u}{\partial x}\right)^{p-1} \\
&= -\int_{\mathbb{R}} \frac{\partial(f'(u))}{\partial x}\left(\frac{\partial u}{\partial x}\right)^p - \frac{1}{p}\int_{\mathbb{R}} f'(u)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)^p \\
&= -\frac{1}{p}\int_{\mathbb{R}} \frac{\partial}{\partial x}\left(f'(u)\left(\frac{\partial u}{\partial x}\right)^p\right) - (1-\frac{1}{p})\int_{\mathbb{R}} f''(u)\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial x}\right)^p \leq 0,
\end{aligned}$$

because f is convex, u verifies $(H2)$ and $p \geq 1$. To terminate the demonstration, we follow the same steps of the proof of Theorem 1.1. We remark that here we do not need the L^2 bound over the solution and also the compactness in the passage to the limit, because the equation (1.4) is in the conservative form which was not the case of our study. □

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Chapitre 3

Convergence d'un schéma pour un système couplé non-local modélisant la dynamique de densités de dislocations

Ce chapitre est une version rallongée d'un travail en collaboration avec N. Forcadel, il s'agit d'un article à paraître dans *Mathematics of Computation.*

Dans ce papier, nous étudions un système couplé non-local qui intervient dans la théorie de la dynamique des densités de dislocations. Dans le cadre des solutions viscosité, nous prouvons un résultat d'existence et d'unicité en temps long pour la solution du modèle. Nous proposons également un schéma numérique et nous montrons une estimation d'erreur de type Crandall-Lions entre la solution continue et son approximation numérique. À notre connaissance, il s'agit de la première estimation de type Crandall-Lions pour un système d'Hamilton-Jacobi. Nous présentons également quelques simulations numériques.

convergent scheme for a non-local coupled system modelling dislocations densities dynamics

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Abstract

In this paper, we study a non-local coupled system that arises in the theory of dislocations densities dynamics. Within the framework of viscosity solutions, we prove a long time existence and uniqueness result for the solution of this model. We also propose a convergent numerical scheme and we prove a Crandall-Lions type error estimate between the continuous solution and the numerical one. As far as we know, this is the first error estimate of Crandall-Lions type for Hamilton-Jacobi systems. We also provide some numerical simulations.

AMS Classification : 35Q72, 49L25, 35F25, 35L40, 65M06, 65M12, 65M15, 74H20, 74H25.

Keywords : Hamilton Jacobi equations, viscosity solutions, dislocations densities dynamics, numerical scheme, error estimate, system.

1 Introduction

1.1 Presentation and physical motivations

A dislocation is a crystal defect which corresponds to a discontinuity in the crystalline structure organisation. This concept has been introduced by Polanyi, Taylor and Orowan in 1934 as the main explanation at the microscopic scale of plastic deformation. A dislocation creates around it a perturbation that can be seen as an elastic field. Under an exterior strain, a dislocation moves according to its Burgers vector which caracterize the intensity and the direction of the defect displacement (see Hirth and Lothe [74] for an introduction to dislocations).

Here, we are interested in dislocations densities dynamics. More precisely, we consider edge dislocations, *i.e* the Burgers vectors and dislocations are in the same plane. These dislocations are moving with the Burgers vectors $\pm\vec{b}$ (see figure 3.1). This model has been introduced by Groma, Balogh as a coupled system, namely a transport problem where the velocity is given by the elasticity equations in the 2-D case (see [71]).

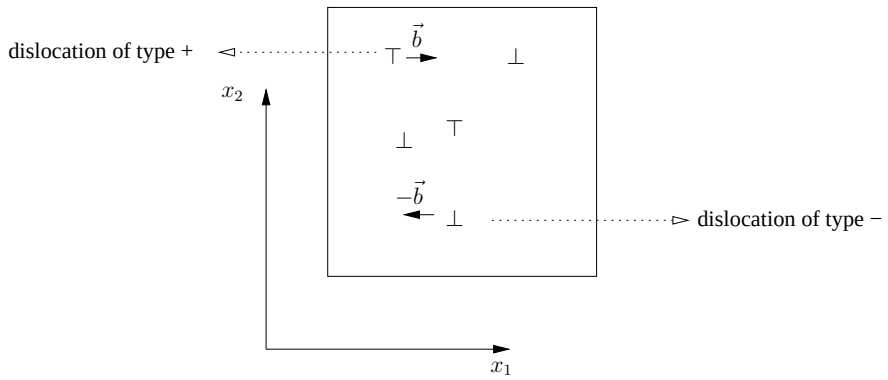

FIG. 3.1 – The cross-section of the dislocations lines.

If the 2-D domain is 1-periodic in x_1 and x_2, and if the dislocations densities depend only on the variable $x = x_1 + x_2$ (where (x_1, x_2) are the coordinates of a point in $\mathbb{R}^2$), when $\vec{b} = (1, 0)$, the 2-D model of [71] reduces to the system of coupled 1-D non local Hamilton-Jacobi equations (see Section 3)

$$\begin{cases} (\rho_+)_t = -\left(\rho_+ - \rho_- + \displaystyle\int_0^1 (\rho_+(x,t) - \rho_-(x,t))\,dx + L(t)\right)|D\rho_+| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on} \quad \mathbb{R}\times(0,T) \\ (\rho_-)_t = \left(\rho_+ - \rho_- + \displaystyle\int_0^1 (\rho_+(x,t) - \rho_-(x,t))\,dx + L(t)\right)|D\rho_-| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on} \quad \mathbb{R}\times(0,T) \end{cases} \tag{1.1}$$

where ρ_+, ρ_- are the unknown scalars such that $(\rho_+ - \rho_-)$ represents the plastic deformation, their space derivatives $D\rho_\pm := \frac{\partial \rho_\pm}{\partial x}$ are the dislocations densities and $L(t)$ represents the exterior shear stress field. From a physical viewpoint, $D\rho_\pm \geq 0$, however, here we do not make this assumption to remain on a more general framework. The initial conditions for the system (1.1) are defined as follows :

$$\rho_\pm(x,0) = \rho_\pm^0(x) = P_\pm^0(x) + L_0 x \text{ on } \mathbb{R} \tag{1.2}$$

where $P_\pm^0$ are periodic of period 1 and Lipschitz continuous. In particular, $\rho_+^0 - \rho_-^0$ is a 1-periodic function. L_0 is a given constant which is the total densities of type $\pm$, *i.e.* we suppose that initially, we have the same total density of type $+$ and $-$.

1.2 Main Results

The first goal of our paper is to prove the existence and uniqueness for the solution of the non-local system (1.1)-(1.2). A natural framework for our study is the viscosity solution theory. We refer to Barles [12], Bardi, Capuzzo-dolcetta [10] and Crandall, Ishii, Lions [37] for a good introduction to this theory in the scalar case. We also refer to Ishii, Koike [83] and Ishii [82] for the vectorial case and to Engler, Lenhart [50], Ishii, Koike [84], Lenhart [101], Lenhart, Belbas [102], Lenhart, Yamada [103] and Yamada [134] for some applications.

We have the following existence and uniqueness result for the non local system :

Theorem 1.1 (Existence and uniqueness for the non-local problem) *For all $T > 0$, for all $L_0 \in \mathbb{R}$, suppose that $\rho^0_\pm \in \mathrm{Lip}(\mathbb{R})$ satisfy (1.2) and $L \in W^{1,\infty}(\mathbb{R}^+)$. Then, the system (1.1)-(1.2) admits a unique viscosity solution $\rho = (\rho_+, \rho_-)$. Moreover, this solution is uniformly Lipschitz continuous in space and time.*

Remark 1.2 *If at initial time, we have $D\rho^0_\pm \geq 0$, then this remains true for $t \geq 0$, i.e., $D\rho_\pm(x,t) \geq 0$ for all $(x,t) \in \mathbb{R} \times [0,T]$. This allows to treat the physical case where $D\rho_\pm \geq 0$.*

The main difficulty comes from the fact that the comparison principle does not hold because of the non-local term. In order to overcome this problem, we classically use a fixed point method by freezing the non-local term. In a first time, we give an existence and uniqueness result for the local problem (this is a simple adaptation of [83]). Then, we use Lipschitz estimates on the solution to prove the short time existence and uniqueness for the non-local system. In the third step, we obtained the result for all time by iterating the process.

Here, we are interested in the dislocations densities dynamics. Some others models have been proposed to describe the dynamics of dislocations lines. We recall some recent results. A non-local Hamilton-Jacobi equation have been proposed by Alvarez, Hoch, Le Bouar and Monneau [7] [6] for modelling dislocation dynamics. They also proved a short time existence and uniqueness result for this model. We also refer to Alvarez, Cardaliaguet, Monneau [3] and Barles, Ley [15] for a long time result under certain monotony assumptions and to Forcadel [55] for a short time result for dislocations dynamics with a mean curvature term.

The second result is a numerical analysis of the non-local system (1.1). We propose a numerical scheme for our non-local system. Then, we give an error estimate between the continuous solution and the numerical one.

We want to approximate the solution of (1.1)-(1.2). Given a mesh size Δx, Δt, we define

$$\Xi = \{i\Delta x,\ i \in \mathbb{Z}\} \quad \Xi_T = \Xi \times \{0, ..., (\Delta t)N_T\}$$

where N_T is the integer part of $T/\Delta t$. We refer generically to the lattice by Δ in the sequel. The discrete running point is (x_i, t_n) with $x_i = i(\Delta x)$, $t_n = n(\Delta t)$. We assume that $\Delta x + \Delta t \leq 1$. The approximation of the solution ρ_k at the node (x_i, t_n) is written indifferently as $v_k(x_i, t_n)$ or $v^n_{k,i}$ according to whether we view it as a function defined on the lattice or as a sequence.

Now, we will introduce the numerical scheme. The main difficulty is due to the non-local term, which requires the availability of the solution we are intending to approximate. To solves this problem, we fix the solution $v^n_i = (v^n_{+,i}, v^n_{-,i})$ at each time step on the interval $[t_n, t_{n+1}]$ and we apply the following monotone scheme,

$$v^0_i = (v^0_{+,i}, v^0_{-,i}) = \tilde{\rho}^0(x_i) = (\tilde{\rho}^0_+, \tilde{\rho}^0_-), \tag{1.3}$$

where $\tilde{\rho}^0_\pm(x_i)$ is an approximation of $\rho^0_\pm(x_i)$; and $\forall k \in \{+, -\}$

$$v^{n+1}_{k,i} = v^n_{k,i} + \Delta t C^\Delta_k[v](x_i, t_n) \begin{cases} E^+\left(D^+ v^n_{k,i}, D^- v^n_{k,i}\right) & \text{if } C^\Delta_k[v](x_i, t_n) \geq 0 \\ E^-\left(D^+ v^n_{k,i}, D^- v^n_{k,i}\right) & \text{if not} \end{cases} \tag{1.4}$$

where

$$C^\Delta_k[v](x_i, t_n) = -k\left(v^n_{+,i} - v^n_{-,i} + a^\Delta[v](t_n)\right)$$

and the non-local term $a^\Delta[v](t_n)$ is given by

$$a^\Delta[v](t_n) = \sum_{i=0}^{N_x - 1} \Delta x \left(v_+(x_i, t_n) - v_-(x_i, t_n)\right) + L(t_n) \tag{1.5}$$

where N_x is the integer part of $1/\Delta x$. $E^\pm$ are the approximation of the Euclidean norm proposed by Osher and Sethian [116] :

$$E^+(P, Q) = \left(\max(P, 0)^2 + \min(Q, 0)^2\right)^{\frac{1}{2}}, \tag{1.6}$$

$$E^-(P, Q) = \left(\min(P, 0)^2 + \max(Q, 0)^2\right)^{\frac{1}{2}}$$

and $D^+ v^n_{k,i}$, $D^- v^n_{k,i}$ are the discrete gradient for all $n \in \{0, ..., N_T\}$, $i \in \mathbb{Z}$ and $k \in \{+, -\}$:

$$D^+ v^n_{k,i} = \frac{v^n_{k,i+1} - v^n_{k,i}}{\Delta x}, \tag{1.7}$$

$$D^- v^n_{k,i} = \frac{v^n_{k,i} - v^n_{k,i-1}}{\Delta x}.$$

Finally, we assume the following uniform CFL condition (see the beginning of Section 5.2 for more details)

$$\Delta t \leq \frac{1}{2L_2} \Delta x \tag{1.8}$$

where

$$L_2 = 2M + 2$$

with $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$.

We then have the following error estimate :

Theorem 1.3 (Discrete-continuous error estimate) *Let $T \geq 0$. Assume that $\Delta x + \Delta t \leq 1$, $L \in W^{1,\infty}(\mathbb{R} \times [0,T))$ and that the CFL condition (1.8) holds. Then there exists a constant $K > 0$ depending only on $\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$, $\|L\|_{W^{1,\infty}(0,T)}$ and $\max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ such that the error estimate between the continuous solution ρ of the system (1.1)-(1.2) and the discrete solution v of the finite difference scheme (1.3)-(1.4) is given by*

$$\max_{k\in\{+,-\}} \sup_{\Xi_T} |\rho_k - v_k| \leq K \left((T + \sqrt{T}) (\Delta x + \Delta t)^{1/2} + \max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0| \right)$$

provided $K \left((T + \sqrt{T})(\Delta x + \Delta t)^{\frac{1}{2}} + \max_{k\in\{+,-\}} \sup_{\Xi} (\rho_k^0 - v_k^0) \right) \leq 1.$

Remark 1.4 *In the condition* $K \left((T + \sqrt{T})(\Delta x + \Delta t)^{\frac{1}{2}} + \max_{k\in\{+,-\}} \sup_{\Xi} (\rho_k^0 - v_k^0) \right) \leq 1$, *we can replace the right hand side by any positive constant.*

In fact in the proof of this theorem, we mimic the continuous problem by considering the approximate solution of (1.1) as a fixed point of a local system. We are inspired by [5] to prove a Crandall-Lions rate of convergence [40], between the continuous solution of (1.1) and the numerical one. As far as we know, this is the first error estimate of Crandall-Lions type for Hamilton-Jacobi systems. We also refer to Jakobsen, Karlsen [86] and Jakobsen, Karlsen, Risebro [87] where they proved an error estimate for a weakly coupled system of the form

$$(u_i)_t + H_i(t, x, u_i, Du_i) = G_i(t, x, u) \quad \text{in } \mathbb{R}^N \times (0, T) \tag{1.9}$$

for $i = 1, ..., M$. Their error estimate is in $O(\Delta t)$ for a semi-discrete splitting algorithm that they propose to approach the solution of (1.9). However, we obtain an error estimate in $O(\sqrt{\Delta t + \Delta x})$ because we also discretize in space.

In the dynamics of dislocations lines case, the model have also been numerically studied by Alvarez, Carlini, Monneau and Rouy [4,5]. In their paper, they proposed a numerical scheme for the non-local Hamilton-Jacobi equation and they proved a Crandall-Lions type rate of convergence.

Let us now explain how the paper is organised. In Section 2, we fix some notations. We present the formal derivation of the model in Section 3. Then, in Section 4, we study the continuous problem. First in Subsection 4.1, we give an existence and

uniqueness result for a local system. Then, in Subsection 4.2, we prove Theorem 1.1 by using a fixed point method. In Section 5, we prove a Crandall-Lions type error estimate for the local problem and then we prove Theorem 1.3 on the non-local problem. Some numerical examples are displayed in Section 6 where we show some tests illustrating our error estimate and then an evolution approximation of dislocation densities.

2 Notation

For simplicity of presentation, we fix some notations :

1. *Order relation* : for $r = (r_1,\ r_2)\,,\ \ s = (s_1,\ s_2) \in \mathbb{R}^2$, we say that $r \leq s$ if $r_k \leq s_k$ for $k \in \{1, 2\}$.
2. *Addition vector-scalar* : for $r = (r_1,\ r_2) \in \mathbb{R}^2,\ \lambda \in \mathbb{R}$, we denote by $r + \lambda$ the vector $(r_1 + \lambda,\ r_2 + \lambda)$.
3. *P-periodic plus L_0-linear function* : we say that ρ is $P-$periodic plus L_0-linear if there exists a vectorial periodic in space function $P^\rho = (P^\rho_+, P^\rho_-)$ of period P and a constant L_0 such that

$$\rho(x,t) = P^\rho(x,t) + L_0 x = (P^\rho_+(x,t) + L_0 x,\ P^\rho_-(x,t) + L_0 x)\,.$$

3 Modelling

We denote by $\mathbf{X}$ the vector $\mathbf{X} = (x_1, x_2)$. We consider a crystal with periodic deformation, namely the case where the total displacement of the crystal $U = (U_1, U_2) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^2$ can be decomposed in a 1-periodic displacement $u = (u_1, u_2)$ and a linear displacement $A(t)^t\mathbf{X}$ with $A(t)$ a given 2×2 matrix which represents the shear stress

$$A(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix}.$$

The displacement U is then given by

$$U(\mathbf{X}, t) = u(\mathbf{X}, t) + A(t)^t\mathbf{X}$$

and we define the total strain by

$$\varepsilon(U) = \frac{1}{2}(\nabla U + {}^t\nabla U) = \frac{1}{2}\left(\nabla u + {}^t\nabla u + A(t) + {}^tA(t)\right).$$

where the coefficients of ∇u are $(\nabla u)_{ij} = \dfrac{\partial u_i}{\partial x_j}$, $i, j \in \{1, 2\}$.
This total strain is decomposed in the form

$$\varepsilon(U) = \varepsilon^e(U) + \varepsilon^p,$$

where $\varepsilon^e(U)$ is the elastic deformation and ε^p the plastic deformation which is connected to the densities of dislocations by

$$\varepsilon^p = \varepsilon^0 \left(\rho_+ - \rho_-\right), \tag{3.10}$$

where $\rho_\pm$ represent the edge dislocation of type $\pm$, such that $\vec{b}.\nabla\rho_\pm \geq 0$ is the density of dislocation of type $\pm$, $\vec{b} = (b_1, b_2)$ is the Burgers vector and

$$\varepsilon^0 = \frac{1}{2}\left(\vec{b} \otimes \vec{b}^\perp + \vec{b}^\perp \otimes \vec{b}\right) \tag{3.11}$$

where $\vec{b}^\perp$ is a vector orthogonal to $\vec{b}$ and $\left(\vec{b} \otimes \vec{b}^\perp\right)_{ij} = b_i b_j^\perp$.

The stress is then given by

$$\sigma = \Lambda : \varepsilon^e(U), \tag{3.12}$$

i.e. the coefficients of the matrix σ are :

$$\sigma_{ij} = \sum_{k,l \in \{1,2\}} \Lambda_{ijkl} \varepsilon^e_{kl}(U) \quad i, j \in \{1, 2\}$$

with $\Lambda = (\Lambda_{ijkl})_{ijkl}$, $i, j, k, l = 1, 2$, Λ_{ijkl} are the elastic constant coefficients of the material, satisfying for $m > 0$:

$$\sum_{ijkl=1,2} \Lambda_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \geq m \sum_{ij=1,2} \varepsilon_{ij}^2 \tag{3.13}$$

for all symmetric matrix $\varepsilon = (\varepsilon_{ij})_{ij}$ *i.e.* such that $\varepsilon_{ij} = \varepsilon_{ji}$.

The functions $\rho_\pm$ and u are then solutions of the coupled system (see Groma, Balogh [71], [70] and Groma [69]) :

$$\begin{cases} \operatorname{div} \sigma &= 0 & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ \sigma &= \Lambda : (\varepsilon(U) - \varepsilon^p) & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ \varepsilon(U) &= \frac{1}{2}\left(\nabla u + {}^t\nabla u + A(t) + {}^tA(t)\right) & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ \varepsilon^p &= \varepsilon^0 \left(\rho_+ - \rho_-\right) & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ \\ (\rho_\pm)_t &= \pm(\sigma : \varepsilon^0)\vec{b}.\nabla\rho_\pm & \text{in } \mathbb{R}^2 \times (0\ \ T), \end{cases} \tag{3.14}$$

i.e. in the coordinates

$$\begin{cases} \displaystyle\sum_{j=1,2} \frac{\partial \sigma_{ij}}{\partial x_j} &= 0 & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ \sigma_{ij} &= \displaystyle\sum_{k,l\in\{1,2\}} \Lambda_{ijkl}\left(\varepsilon_{kl}(U) - \varepsilon^p_{kl}\right) & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ \varepsilon_{ij}(U) &= \frac{1}{2}\Big((\nabla u)_{ij} + (\nabla u)_{ij} + A_{ij}(t) + A_{ji}(t)\Big) & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ \varepsilon^p_{ij} &= \varepsilon^0_{ij}\left(\rho_+ - \rho_-\right) & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ (\rho_\pm)_t &= \pm\left(\displaystyle\sum_{i,j\in\{1,2\}} \sigma_{ij}\varepsilon^0_{ij}\right)\vec{b}.\nabla\rho_\pm & \text{in } \mathbb{R}^2 \times (0\ \ T), \end{cases} \tag{3.15}$$

where the unknowns of the system are $\rho_\pm$ and the displacement $u = (u_1, u_2)$ and with ε^0 defined by (3.11). The sign $\pm$ comes from $\pm$ in $\pm\vec{b}$.

To simplify, we consider the homogeneous case. The coefficients Λ_{ijkl} are such that

$$\sigma = 2\mu\varepsilon^e(U) + \lambda tr(\varepsilon^e(U))I_d, \tag{3.16}$$

where $\mu > 0$ and $\lambda + \mu > 0$ (consequence of (3.13)) are the Lamé coefficients and I_d the identity matrix. Then, the following *lemma* holds :

***Lemma* 3.1 (Equivalence between 2-D and 1-D models)** *If we assume that the Burger vector is $\vec{b} = (1, 0)$, and that the densities of dislocations and u only depend on one variable $x = x_1 + x_2$ (as shown in Figure 3.2), the 2-D problem (3.14), with Λ defined by (3.16) is equivalent to the 1-D problem*

$$\begin{cases} (\rho_+)_t &= -C_1\left((\rho_+ - \rho_-) + C_2\displaystyle\int_0^1 (\rho_+ - \rho_-) + L(t)\right) D\rho_+ & \textit{in } \mathbb{R} \times (0\ \ T) \\ (\rho_-)_t &= C_1\left((\rho_+ - \rho_-) + C_2\displaystyle\int_0^1 (\rho_+ - \rho_-) + L(t)\right) D\rho_- & \textit{in } \mathbb{R} \times (0\ \ T) \end{cases} \tag{3.17}$$

where $L(t) = -\frac{(\lambda+2\mu)}{(\lambda+\mu)}(A_{12}(t) + A_{21}(t))$, $C_1 = \frac{(\lambda+\mu)\mu}{\lambda+2\mu}$, and $C_2 = \frac{\mu}{(\lambda+\mu)}$.

Proof of lemma 3.1

We can rewrite the first equation of (3.14) and (3.16) as

$$\operatorname{div}\left(2\mu\varepsilon(U) + \lambda tr(\varepsilon(U))I_d\right) = \operatorname{div}\left(2\mu\varepsilon^p + \lambda tr(\varepsilon^p)I_d\right).$$

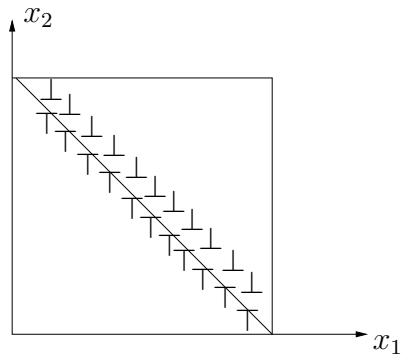

FIG. 3.2 – 1D sub-model for invariance by translation in the direction (-1,1)

This implies by (3.10)

$$\mu\Delta u + (\lambda+\mu)\nabla(\text{div } u) = \mu \begin{pmatrix} \dfrac{\partial(\rho_+ - \rho_-)}{\partial x_2} \\ \\ \dfrac{\partial(\rho_+ - \rho_-)}{\partial x_1} \end{pmatrix}.$$

Using the fact that $x = x_1 + x_2$, yields

$$2\mu \begin{pmatrix} \dfrac{\partial^2 u_1}{\partial x^2} \\ \\ \dfrac{\partial^2 u_2}{\partial x^2} \end{pmatrix} + (\lambda+\mu) \begin{pmatrix} \dfrac{\partial^2 (u_1+u_2)}{\partial x^2} \\ \\ \dfrac{\partial^2 (u_1+u_2)}{\partial x^2} \end{pmatrix} = \mu \begin{pmatrix} \dfrac{\partial(\rho_+ - \rho_-)}{\partial x} \\ \\ \dfrac{\partial(\rho_+ - \rho_-)}{\partial x} \end{pmatrix}.$$

Now, by adding the two above equations, we obtain

$$\frac{\partial^2(u_1+u_2)}{\partial x^2} = \frac{\mu}{\lambda+2\mu}\left(\frac{\partial(\rho_+ - \rho_-)}{\partial x}\right).$$

Integrating the above equation yields, since u is 1-periodic :

$$\frac{\partial(u_1+u_2)}{\partial x} = \frac{\mu}{\lambda+2\mu}\left((\rho_+ - \rho_-) - \int_0^1 (\rho_+ - \rho_-)\right). \tag{3.18}$$

Using the fact that

$$(\sigma : \varepsilon^0) = \sigma_{12} = 2\mu(\varepsilon^e(U))_{12} = \mu\left(\frac{\partial(u_1+u_2)}{\partial x} + A_{12}(t) + A_{21}(t) - (\rho_+ - \rho_-)\right)$$

and (3.18) yields

$$(\sigma : \varepsilon^0) = -\frac{(\lambda+\mu)\mu}{\lambda+2\mu}\left((\rho_+ - \rho_-) + \frac{\mu}{2(\lambda+\mu)}\int_0^1 (\rho_+ - \rho_-) + L(t)\right)$$

where $L(t) = -\dfrac{(\lambda+2\mu)}{(\lambda+\mu)}(A_{12}(t)+A_{21}(t))$. We then deduce, if $\vec{b} = (1,0)$, that the system (3.14) can be rewritten as (3.17). As the constant C_1, C_2 are positive, to simplify the notations, we can put them to 1 in the following without lost of generality on the system (1.1). □

4 The continuous problem

To prove the existence and uniqueness result for the non-local problem, we use a fixed point method. In order to do that we freeze the non-local term and we study the following local problem

$$\begin{cases} (\rho_+)_t = -\left(\rho_+ - \rho_- + a(t)\right)|D\rho_+| & \text{on} \quad \mathbb{R}\times(0,T), \\ (\rho_-)_t = \left(\rho_+ - \rho_- + a(t)\right)|D\rho_-| & \text{on} \quad \mathbb{R}\times(0,T), \\ \rho_+(\cdot,0) = \rho_+^0 & \text{on} \quad \mathbb{R}, \\ \rho_-(\cdot,0) = \rho_-^0 & \text{on} \quad \mathbb{R}. \end{cases} \tag{4.19}$$

The assumptions are the following :

(H1) $a \in W^{1,\infty}(\mathbb{R}^+)$,

(H2) $\rho^0 = (\rho_+^0, \rho_-^0)$ is $1-$periodic plus L_0-linear, *i.e.*, $\rho_\pm^0(x) = P_\pm^0(x) + L_0 x$ where $P_\pm^0$ are periodic of period 1 and L_0 is a constant.

(H3) $P_\pm^0 \in \mathrm{Lip}(\mathbb{R})$.

4.1 The local problem

In this subsection, we will show some existence and uniqueness results for the local Hamilton-Jacobi system (4.19) within the framework of viscosity solution. We denote by $USC(\mathbb{R}\times(0,T))$ (resp. $LSC(\mathbb{R}\times(0,T))$) the set of locally bounded upper (resp. lower) semi-continuous functions. For $k \in \{+,-\}$, we define the following Hamiltonian

$$H_k(t,\rho,p) = k(\rho_+ - \rho_- + a(t))|p|.$$

We recall the definition of viscosity solution for Problem (4.19), proposed by Ishii, Koike [83].

Definition 4.1 (Subsolutions, supersolutions, solutions) *: A function $\rho \in USC(\mathbb{R}\times]0,T[)$ is a viscosity subsolution of (4.19), if*

- $\rho(\cdot, t=0) \le \rho_0$
- *for all $k \in \{+,-\}$ and for any test-function $\phi \in C^1(\mathbb{R}\times]0,T[)$ such that $\rho_k - \phi$ reaches a local maximum at a point $(x_0,t_0) \in \mathbb{R}\times]0,T[$, we have*

$$\phi_t(x_0,t_0) + H_k(t_0,\rho(x_0,t_0),D\phi(x_0,t_0)) \le 0$$

In a similar way, a function $\rho \in LSC(\mathbb{R}\times]0,T[)$ is a viscosity supersolution of (4.19) if

- $\rho(\cdot, t=0) \geq \rho_0$
- *for all $k \in \{+,-\}$ and for any test-function $\phi \in C^1(\mathbb{R}\times]0,T[)$ such that $\rho_k - \phi$ reaches a local minimmum at a point $(x_0,t_0) \in \mathbb{R}\times]0,T[$, we have*

$$\phi_t(x_0,t_0) + H_k(t_0, \rho(x_0,t_0), D\phi(x_0,t_0)) \geq 0$$

Finally, ρ is a viscosity solution of (4.19) if and only if ρ is sub- and supersolution of (4.19).

The key point is that our system is quasi monotone in the sense of Ishii, Koike [83, (A.1)], (see *Lemma* 4.2 below) and so we can extend their results to our system in unbounded domain and with unbounded initial condition using the well-known arguments of the scalar case.

***Lemma* 4.2 (Quasi-monotony of the Hamiltonian)**
The Hamiltonian H is quasi-monotone, i.e., for all vectors r and s such that

$$r_j - s_j = \max_{k\in\{+,-\}}(r_k - s_k) \geq 0$$

then

$$H_j^1(t,r,p) - H_j^1(t,s,p) \geq 0. \tag{4.20}$$

Proof of Lemma 4.2 :
Let r and s be two vectors such that $r_j - s_j = \max_{k\in\{+,-\}}(r_k - s_k) \geq 0$. We have

$$\begin{aligned} H_j(t,r,p) - H_j(t,s,p) &= j(r_+ - r_- + a(t))|p| - j(s_+ - s_- + a(t))|p| \\ &= j|p|((r_+ - s_+) - (r_- - s_-)) \\ &= |p| sign((r_+ - s_+) - (r_- - s_-))((r_+ - s_+) - (r_- - s_-)) \\ &= |p||(r_+ - s_+) - (r_- - s_-)| \geq 0. \end{aligned}$$

This ends the proof. □

Proposition 4.3 (Comparison principle)
Let $\rho \in USC(\Omega\times]0,T[)$ and $v \in LSC(\Omega\times]0,T[)$ be respectively sub and supersolutions of (1.1). We assume that there exists $C > 0$ such that

$$\rho_0(x) - Ct \leq \rho, v \leq \rho_0(x) + Ct. \tag{4.21}$$

Then if $\rho(\cdot,0) \leq v(\cdot,0)$ in $\mathbb{R}$ then $\rho \leq v$ in $\mathbb{R}\times]0,T[$.

Proof of Proposition 4.3
Let us denote by $M_{\sup} = \max_k \sup_{\mathbb{R}\times[0,T]} (\rho_k - v_k)$. It is sufficient to prove that this maximum is non positive. Let us suppose by contradiction the positivity of $M_{\sup}$. We duplicate the variables by considering, for all ε, β, η and α positive

$$\psi(x,y,t,s,k) = \rho_k(x,t) - v_k(y,s) - \frac{|x-y|^2}{2\varepsilon} - \frac{|t-s|^2}{2\beta} - \frac{\eta}{T-t} - \alpha(|x|^2+|y|^2).$$

We note that $\psi(x,y,t,s,k)$ is USC in $(\mathbb{R}\times[0,T[)^2$, because of the term $\frac{|x-y|^2}{2\varepsilon}+\frac{|t-s|^2}{2\beta}$. We can think that the maximum of ψ noted $M(\varepsilon,\beta,\alpha,\eta) = \sup_{x,y,t,s,k} \psi(x,y,t,s,k)$, is similar with $M_{\sup}$.
This idea is justify by the following lemma

***Lemma* 4.4** *: Let $(\bar{x},\bar{y},\bar{t},\bar{s},\bar{k})$ be a maximum of ψ. If we define $M' = \lim_{h\to 0} M_h$, where $M_h = \sup_{\substack{|x-y|\le h\\ |t-s|\le h}} (\rho(x,t) - v(y,s))$ Then the following properties hold*

1. $\lim_{\alpha\to 0} \alpha|\bar{x}| = \lim_{\alpha\to 0} \alpha|\bar{y}| = 0$
2. $\limsup_{(\varepsilon,\beta,\alpha,\eta)\to 0} M(\varepsilon,\beta,\alpha,\eta) = M'$
3. $\limsup_{(\varepsilon,\beta,\alpha,\eta)\to 0} u_{\bar{k}}(\bar{x},\bar{t}) - v_{\bar{k}}(\bar{y},\bar{s}) = M'$
4. $\frac{|\bar{x}-\bar{y}|^2}{2\varepsilon} \to 0$ *and* $\frac{|\bar{t}-\bar{s}|^2}{2\beta} \to 0$ *when* $(\varepsilon,\beta,\alpha,\eta)\to 0$
5. $\bar{t},\bar{s}$ *are positive if* ε, β, α *and* η *are sufficiently small.*

The proof is postponed.

We take ε, β, α and η small enough such that $\bar{t}>0$ and $\bar{s}>0$ (see Lemma 4.4). Using the fact that ρ and v are sublinear, we deduce that

$$\liminf_{|x|,|y|\to\infty} \psi(x,y,t,s) = -\infty$$

and so the function ψ reaches a maximum at a point $(\bar{x},\bar{y},\bar{t},\bar{s},\bar{k})$. We then deduce that the function

$$(x,t) \to \rho_{\bar{k}}(x,t) - \left[v_{\bar{k}}(\bar{y},\bar{s}) + \frac{|x-\bar{y}|^2}{2\varepsilon} + \frac{|t-\bar{s}|^2}{2\beta} + \frac{\eta}{T-t} + \alpha(|x|^2+|\bar{y}|^2)\right]$$

reaches a maximum at $(\bar{x},\bar{t})$. By using the test-function

$$\phi(x,t) = v_{\bar{k}}(\bar{y},\bar{s}) + \frac{|x-\bar{y}|^2}{2\varepsilon} + \frac{|t-\bar{s}|^2}{2\beta} + \frac{\eta}{T-t} + \alpha(|x|^2+|\bar{y}|^2)$$

and the fact that ρ is a subsolution of (4.19), we deduce

$$\frac{(\bar{t}-\bar{s})}{\beta}+\frac{\eta}{(T-\bar{t})^2}+H_{\bar{k}}\left(\bar{t},u(\bar{x},\bar{t}),\frac{(\bar{x}-\bar{y})}{\varepsilon}+2\alpha\bar{x}\right)\leq 0. \tag{4.22}$$

In the same way, we have

$$\frac{(\bar{t}-\bar{s})}{\beta}+H_{\bar{k}}\left(\bar{s},v(\bar{y},\bar{s}),\frac{(\bar{x}-\bar{y})}{\varepsilon}-2\alpha\bar{y}\right)\geq 0. \tag{4.23}$$

By subtracting (4.23) to (4.22), we deduce

$$\frac{\eta}{(T-\bar{t})^2}+H_{\bar{k}}\left(\bar{t},\rho(\bar{x},\bar{t}),\frac{(\bar{x}-\bar{y})}{\varepsilon}+2\alpha\bar{x}\right)-H_{\bar{k}}\left(\bar{s},v(\bar{y},\bar{s}),\frac{(\bar{x}-\bar{y})}{\varepsilon}-2\alpha\bar{y}\right)\leq 0. \tag{4.24}$$

By using Lemma 4.4, we have, up to extract a subsequence

$$\lim_{\beta\to 0}\bar{x}=\tilde{x},\quad \lim_{\beta\to 0}\bar{y}=\tilde{y},$$
$$\lim_{\beta\to 0}\bar{t}=\lim_{\beta\to 0}\bar{s}=\tau.$$

Sending β to 0 in (4.24), we deduce

$$\frac{\eta}{(T-\tau)^2}+H_{\tilde{k}}\left(\tau,\rho(\tilde{x},\tau),\frac{(\tilde{x}-\tilde{y})}{\varepsilon}+2\alpha\tilde{x}\right)-H_{\tilde{k}}\left(\tau,v(\tilde{y},\tau),\frac{(\tilde{x}-\tilde{y})}{\varepsilon}-2\alpha\tilde{y}\right)\leq 0. \tag{4.25}$$

with $\tilde{k}=\lim_{\beta\to 0}\bar{k}$.

Moreover, we have

$$\rho_{\tilde{k}}(\tilde{x},\tau)-v_{\tilde{k}}(\tilde{y},\tau)\geq\rho_k(\tilde{x},\tau)-v_k(\tilde{y},\tau)\geq 0.$$

By adding and by subtracting the term $H_{\tilde{k}}\left(\tau,\rho(\tilde{x},\tau),\frac{(\tilde{x}-\tilde{y})}{\varepsilon}-2\alpha\tilde{y}\right)$ in the inequality (4.25) and by using Lemma 4.2, we deduce that

$$\frac{\eta}{(T-\tau)^2}+H_{\tilde{k}}\left(\tau,\rho(\tilde{x},\tau),\frac{(\tilde{x}-\tilde{y})}{\varepsilon}+2\alpha\tilde{x}\right)-H_{\tilde{k}}\left(\tau,\rho(\tilde{x},\tau),\frac{(\tilde{x}-\tilde{y})}{\varepsilon}-2\alpha\tilde{y}\right)\leq 0$$

and so

$$\frac{\eta}{(T-\tau)^2}+\tilde{k}(\rho^+(\tilde{x},\tau)-\rho^-(\tilde{x},\tau)+a(\tau))\left(\left|\frac{(\tilde{x}-\tilde{y})}{\varepsilon}+2\alpha\tilde{x}\right|-\left|\frac{(\tilde{x}-\tilde{y})}{\varepsilon}-2\alpha\tilde{y}\right|\right)\leq 0.$$

Using the fact that $\tilde{k}((\rho^+(\tilde{x},\tau)-\rho^-(\tilde{x},\tau)+a(\tau))$ is bounded (see (4.21)), and sending $\alpha\to 0$, we obtain using Lemma 4.4

$$\frac{\eta}{(T-\tau)^2}\leq 0,$$

this contradiction ends the proof of the proposition. □

Proof of lemma 4.4

Thanks to the positivity of $M_{\sup}$, we have

$$M(\varepsilon, \beta, \alpha, \eta) > 0$$

for η and α small enough. We then deduce

$$\begin{aligned}
\alpha(|\bar{x}|^2 + |\bar{y}^2|) &\leq \rho_{\bar{k}}(\bar{x}, \bar{t}) - v_{\bar{k}}(\bar{y}, \bar{s}) - \frac{|\bar{x} - \bar{y}|^2}{2\varepsilon} \\
&\leq \rho_{\bar{k},0}(\bar{x}) - \rho_{\bar{k},0}(\bar{y}) + C(\bar{t} + \bar{s}) - \frac{|\bar{x} - \bar{y}|^2}{2\varepsilon} \\
&\leq K(1 + |\bar{x} - \bar{y}|) - \frac{|\bar{x} - \bar{y}|^2}{2\varepsilon} \\
&\leq K + \tfrac{1}{2}(K^2 + |\bar{x} - \bar{y}|^2) - \frac{|\bar{x} - \bar{y}|^2}{2\varepsilon} \\
&\leq K_1
\end{aligned}$$

for $\varepsilon \leq 1$, where we have used (4.21) for the second line. Multiplying the previous inequality by α, yields

$$\lim_{\alpha \to 0} \alpha|\bar{x}| = \lim_{\alpha \to 0} \alpha|\bar{y}| = 0. \tag{4.26}$$

In the same way, we have

$$\frac{|\bar{x} - \bar{y}|^2}{4\varepsilon} + \frac{|\bar{t} - \bar{s}|^2}{2\beta} \leq K_1 \tag{4.27}$$

and so

$$\lim_{\varepsilon \to 0} |\bar{x} - \bar{y}|^2 = \lim_{\beta \to 0} |\bar{t} - \bar{s}|^2 = 0. \tag{4.28}$$

We recall that $M_h = \sup\limits_{\substack{|x-y| \leq h \\ |t-s| \leq h}} (\rho(x,t) - v(y,s))$. Let $(x_n^h, y_n^h, t_n^h, s_n^h)$ be such that

$$\rho(x_n^h, t_n^h) - v(y_n^h, s_n^h) \geq M_h - \frac{1}{n}$$

with $|x_n^h - y_n^h| \leq h$ and $|t_n^h - s_n^h| \leq h$. We then have

$$\begin{aligned}
&M_h - \frac{1}{n} - \frac{h^2}{2\varepsilon} - \frac{h^2}{2\beta} - \alpha\left(|x_n^h|^2 + |y_n^h|^2\right) \\
\leq &\rho(x_n^h, t_n^h) - v(y_n^h, s_n^h) - \frac{|x_n^h - y_n^h|^2}{2\varepsilon} - \frac{|t_n^h - s_n^h|^2}{2\beta} - \alpha\left(|x_n^h|^2 + |y_n^h|^2\right) \\
\leq &M(\varepsilon, \beta, \alpha, \eta) \\
\leq &\rho(\bar{x}, \bar{t}) - v(\bar{y}, \bar{s}).
\end{aligned}$$

We send $\beta \to 0$ and then $\alpha \to 0$ and we obtain

$$\begin{aligned} M_h - \frac{1}{n} - \frac{h^2}{2\varepsilon} - \frac{h^2}{2\beta} &\leq \liminf_{\alpha\to 0} \liminf_{\beta\to 0} \rho(\bar{x},\bar{t}) - v(\bar{y},\bar{s}) \\ &\leq \limsup_{\alpha\to 0} \limsup_{\beta\to 0} \rho(\bar{x},\bar{t}) - v(\bar{y},\bar{s}). \end{aligned}$$

Passing to the limit in h, yields

$$\begin{aligned} M' - \frac{1}{n} &\leq \liminf_{\alpha\to 0} \liminf_{\beta\to 0} \rho(\bar{x},\bar{t}) - v(\bar{y},\bar{s}) \\ &\leq \limsup_{\alpha\to 0} \limsup_{\beta\to 0} \rho(\bar{x},\bar{t}) - v(\bar{y},\bar{s}). \end{aligned}$$

We then take $\lim\limits_{\varepsilon\to 0}$ and get

$$\begin{aligned} M' - \frac{1}{n} &\leq \liminf_{\varepsilon\to 0} \liminf_{\alpha\to 0} \liminf_{\beta\to 0} \rho(\bar{x},\bar{t}) - v(\bar{y},\bar{s}) \\ &\leq \limsup_{\varepsilon\to 0} \limsup_{\alpha\to 0} \limsup_{\beta\to 0} \rho(\bar{x},\bar{t}) - v(\bar{y},\bar{s}) \\ &\leq \limsup_{\varepsilon\to 0} \limsup_{\alpha\to 0} \left(\sup_{\substack{|x-y|\leq K\sqrt{\varepsilon} \\ |t-s|\leq K\sqrt{\beta}}} \rho(x,t) - v(y,s) \right) \\ &\leq \lim_{h\to 0} \sup_{\substack{|x-y|\leq h \\ |t-s|\leq h}} \rho(x,t) - v(y,s) \\ &\leq M'. \end{aligned}$$

We then deduce that

$$\liminf_{\varepsilon\to 0} \liminf_{\alpha\to 0} \liminf_{\beta\to 0} \rho(\bar{x},\bar{t}) - v(\bar{y},\bar{s}) = \limsup_{\varepsilon\to 0} \limsup_{\alpha\to 0} \limsup_{\beta\to 0} \rho(\bar{x},\bar{t}) - v(\bar{y},\bar{s}) = M'.$$

In the same way, we get

$$\liminf_{\varepsilon\to 0} \liminf_{\alpha\to 0} \liminf_{\beta\to 0} M(\varepsilon,\beta,\alpha,\eta) = \limsup_{\varepsilon\to 0} \limsup_{\alpha\to 0} \limsup_{\beta\to 0} M(\varepsilon,\beta,\alpha,\eta) = M'$$

and we deduce that

$$\limsup_{\varepsilon\to 0} \limsup_{\alpha\to 0} \limsup_{\beta\to 0} \left(\frac{|\bar{x}-\bar{y}|^2}{2\varepsilon} + \frac{|\bar{t}-\bar{s}|^2}{2\beta} \right) = 0$$

Finally, let us suppose, for example, that $\bar{t} = 0$, then, because of $u_0 \leq v_0$, we have

$$\begin{aligned} u_{\bar{k}}(\bar{x},0) - v_{\bar{k}}(\bar{y},\bar{s}) &\leq u_{\bar{k}}(\bar{x},0) - v_{\bar{k}}(\bar{x},0) + v_{\bar{k}}(\bar{x},0) - v_{\bar{k}}(\bar{y},\bar{s}) \\ &\leq v_{\bar{k}}(\bar{x},0) - v_{\bar{k}}(\bar{y},\bar{s}). \end{aligned}$$

However $M' > 0$ and when ε, β, α and η are sufficiently small, we get a contradiction. When $\bar{s} = 0$, a similar proof could be given which ends the proof of the Lemma 4.4. □

We now prove existence for the problem (4.19). We use the Perron's method for systems proved by Ishii and Koike in [83]. We then have the following theorem

Theorem 4.5 (Existence for the local problem)
Assume (H1)-(H2)-(H3), then there exists a unique viscosity solution ρ of problem (4.19). Moreover, the solution satisfies

$$\rho_{\pm,0}(x) - (M + \|a\|_{L^\infty})B_0 t \leq \rho_\pm(x,t) \leq \rho_{\pm,0}(x) + (M + \|a\|_{L^\infty})B_0 t, \tag{4.29}$$

where $M = \|P_{+,0} - P_{-,0}\|_{L^\infty}$ and $B_0 = \|D\rho_0\|_{L^\infty}$.

Proof of Theorem 4.5
By Perron's method, it suffices to construct viscosity sub and supersolution of (4.19). We claim that $\overline{\rho} = \rho_0 + (M + \|a\|_{L^\infty})B_0 t$ and $\underline{\rho} = \rho_0 - (M + \|a\|_{L^\infty})B_0 t$ are respectively super and subsolution. Indeed, formally

$$\begin{aligned} -k\left(\overline{\rho}_+ - \overline{\rho}_- + a(t)\right)|D\overline{\rho}_k| &\leq \left|P_+^0 - P_-^0 + a(t)\right| |D\rho_{k,0}| \\ &\leq \left(\|P_+^0 - P_-^0\|_{L^\infty} + \|a\|_{L^\infty}\right)\|D\rho_0\|_{L^\infty} \\ &\leq (M + \|a\|_{L^\infty})B_0 = (\bar{\rho}_k)_t. \end{aligned}$$

The proof for the subsolution is exactly the same and we skip it. To achieve the proof, it suffices to use the comparison principle to obtain the uniqueness. □

Proposition 4.6 (Regularity of the solution)
The solution ρ of (4.19) is Lipschitz continuous in space and time. More precisely, ρ satisfies :

$$\|D\rho_\pm\|_{L^\infty} \leq B_0 \tag{4.30}$$

and

$$\|(\rho_\pm)_t\|_{L^\infty} \leq (L_a T + M + \|a\|_{L^\infty})B_0, \tag{4.31}$$

where L_a is the Lipschitz constant of a, $B_0 = \|D\rho_0\|_{L^\infty}$ and $M = \|\rho_{+,0} - \rho_{-,0}\|_{L^\infty}$.

Proof of Proposition 4.6
We set $\rho^h_\pm(x,t) = \rho_\pm(x+h,t)$. Since Problem (4.19) is invariant in space, ρ^h and $\rho^h + \|\rho_{\pm,0}(\cdot) - \rho_{\pm,0}(\cdot + h)\|_{L^\infty}$ are still solutions . Using comparison principle, yields

$$\begin{aligned}|\rho_\pm - \rho^h_\pm| \leq & \|\rho_{\pm,0}(\cdot) - \rho_{\pm,0}(\cdot + h)\|_{L^\infty} \\ \leq & B_0 h.\end{aligned}$$

So, ρ is Lipschitz continuous in space.

For the estimate in time, we consider $v(x,t) = \rho(x, t+h)$. It is easy to check that v is a supersolution of

$$\left\{ \begin{array}{l} (v_+)_t = -\left(v_+ - v_- + a(t)\right)|Dv_+| - L_a B_0 h \\ (v_-)_t = \left(v_+ - v_- + a(t)\right)|Dv_-| - L_a B_0 h. \end{array} \right. \tag{4.32}$$

Indeed, formally

$$\begin{aligned}(v_k)_t(x,t) =& (\rho_k)_t(x,t+h) \\ =& -k\left(\rho_+(x,t+h) - \rho_-(x,t+h) + a(t+h)\right)|D\rho_k(x,t+h)| \\ =& -k\left(v_+(x,t) - v_-(x,t) + a(t)\right)|Dv_k(x,t)| + k(a(t) - a(t+h)||Dv_k| \\ \geq& -k\left(v_+(x,t) - v_-(x,t) + a(t)\right)|Dv_k(x,t)| - L_a h B_0.\end{aligned}$$

Moreover, $\tilde{\rho} = \rho - L_a B_0 h t - (M + \|a\|_{L^\infty})B_0 h$ is a solution of the same equation and satisfies $v(\cdot,0) \geq \tilde{\rho}(\cdot,0)$ (see Theorem 4.5). So, by comparison principle for (4.32), we deduce that

$$\rho(x,t) - \rho(x,t+h) \leq (L_a t + M + \|a\|_{L^\infty})B_0 h \leq (L_a T + M + \|a\|_{L^\infty})B_0 h.$$

Using the same arguments with $\rho(x,t-h)$, we deduce that ρ is Lipshitz continuous in time. □

Proposition 4.7 (Caracterization of the solution)
The solution ρ is $(1, L_0)-$periodic plus linear, i.e.

$$\rho = \begin{pmatrix} P^\rho_+ + L_0 x \\ P^\rho_- + L_0 x \end{pmatrix},$$

where the linear part L_0 is the same of the one of ρ_0 and the period of $P^\rho_\pm$ is 1.

Proof of Poposition 4.7
We set $P^\rho_k = \rho_k - L_0 x$. It suffices to show that P^ρ_k are periodic of period 1. The vector function P^ρ satisfies

$$\left\{ \begin{array}{l} (P^\rho_+)_t = -\left(P^\rho_+ - P^\rho_- + a(t)\right)|DP^\rho_+ + L_0| \\ (P^\rho_-)_t = \left(P^\rho_+ - P^\rho_- + a(t)\right)|DP^\rho_- + L_0| \\ P^\rho_+(\cdot,0) = P^0_+ \\ P^\rho_-(\cdot,0) = P^0_-. \end{array} \right.$$

We then set $v(x,t) = P^\rho(x+1,t)$. By the periodicity of $P^0_\pm$, we obtain that v satisfies the same problem as P^ρ and so, by uniqueness, $v = P^\rho$. This achieves the proof of the proposition. □

Finally, we proved the following theorem :

Theorem 4.8 (The local problem) *Let $T \geq 0$. Assume (H1)-(H2)-(H3). We set $M = \|P^0_+ - P^0_-\|_{L^\infty(\mathbb{R})}$ and $B_0 = \max_{k\in\{+,-\}} \|D\rho^0_k\|_{L^\infty(\mathbb{R})}$. Then, the following holds :*

(i) **Comparison principle.** *Let $\rho \in USC(\mathbb{R}\times(0,T))$ and $v \in LSC(\mathbb{R}\times(0,T))$ be respectively sub and super-solution of (1.1)-(1.2). We assume that there exists a constant $C > 0$ such that* (4.21) *holds. If $\rho(\cdot,0) \leq v(\cdot,0)$* in $\mathbb{R}$ *then $\rho \leq v$ in $\mathbb{R}\times[0,T]$.*

(ii) **Existence.** *There exists a unique viscosity solution ρ of problem (4.19) satisfying* (4.29)*. Moreover, the solution is $1-$periodic plus L_0-linear.*

(iii) **Regularity.** *The solution ρ of (4.19) is Lipschitz continuous in space and time and satisfies*

$$\max_{k\in\{+,-\}} \|D\rho_k\|_{L^\infty(\mathbb{R}\times(0,T))} \leq B_0,$$

$$\max_{k\in\{+,-\}} \|(\rho_k)_t\|_{L^\infty(\mathbb{R}\times(0,T))} \leq B_0(M + \|a\|_{L^\infty(0,T)}).$$

(iv) **Estimate on the solution.** *The solution ρ satisfies*

$$\|\rho_+ - \rho_-\|_{L^\infty(\mathbb{R}\times(0,T))} \leq \|\rho^0_+ - \rho^0_-\|_{L^\infty(\mathbb{R})}.$$

Proof of Theorem 4.8

The comparison principle is just an extension of the one of Ishii, Koike [83, Th 4.7] for quasi-monotone Hamiltonians. For the existence, it suffices to use Perron's method by remarking that $\rho \pm (M + \|a\|_{L^\infty(0,T)})B_0 t$ are resp. super and sub-solution of (4.19). The fact that ρ is $1-$periodic plus L_0-linear comes from the fact that $\rho(x+1,t) + L_0$ is also solution of (iv).

The Lipschitz estimate in space comes from the fact that Problem (4.19) is invariant by space translation. To obtain the Lipschitz estimate in time, it is sufficient to bound the velocity using (4.29).

We now prove (iv). We set

$$m_+(t) = \sup_{x\in(0,1)} \rho_+(x,t) \quad \text{and} \quad m_-(t) = \inf_{x\in(0,1)} \rho_-(x,t).$$

It is easy to chech that m_+ (resp. m_-) is subsolution (resp. supersolution) of $u_t = 0$ which implies the upper bound of (iv). The lower bound is proved similarly. This ends the proof of the theorem. □

4.2 The non-local problem

Before to prove Theorem 1.1, we need the following *lemma* :

***Lemma* 4.9 (Stability of the solution with respect to the velocity)** *Let $T \geq 0$. We consider for $i = 1, 2$ two different equations*

$$\begin{cases} (\rho_k^i)_t = -k\left(\rho_+^i - \rho_-^i + a_i(t)\right)|D\rho_k^i| & \text{for} \quad k \in \{+,-\} \\ \rho_k^i(\cdot, 0) = \rho_k^0 & \text{for} \quad k \in \{+,-\} \end{cases} \tag{4.33}$$

where the coefficients a_i satisfy $(H1)$ and the initial conditions $\rho^0 = (\rho_+^0, \rho_-^0)$ satisfy (H2)-(H3). Then, we have

$$\max_{k\in\{+,-\}} \|\rho_k^2 - \rho_k^1\|_{L^\infty(\mathbb{R}\times(0,T))} \leq B_0 T \|a_2 - a_1\|_{L^\infty(0,T)},$$

where ρ^i for $i = 1, 2$ are the solutions of (4.33) given by Theorem 4.8.

Proof of *Lemma* 4.9
We set $K = \|a_2 - a_1\|_{L^\infty(0,T)}$. We remark that ρ^2 is a sub-solution of

$$(\rho_k)_t + k\left(\rho_+ - \rho_- + a_1(t)\right)|D\rho_k| - KB_0 = 0.$$

Moreover $\rho^1 + KB_0 t$ is solution of the same problem. By comparison principle, we then deduce

$$\max_{k\in\{+,-\}} \|\rho_k^2 - \rho_k^1\|_{L^\infty(\mathbb{R}\times(0,T))} \leq KB_0 T.$$

This is the estimate we want. □

We have the following *lemma* whose proof is trivial :

***Lemma* 4.10 (Stability of the velocity a)** *Let ρ^1, ρ^2 be 1-periodic plus L_0-linear. We set $a[\rho^i](t) = \displaystyle\int_0^1 \rho_+^i(x,t) - \rho_-^i(x,t)dx + L(t)$. Then the following holds*

$$\|a[\rho^2] - a[\rho^1]\|_{L^\infty(0,T)} \leq 2 \max_{k\in\{+,-\}} \|\rho_k^2 - \rho_k^1\|_{L^\infty(\mathbb{R}\times(0,T))}.$$

We now prove Theorem 1.1.

Proof of Theorem 1.1
We define the set :

$$U_T = \left\{ \rho = \begin{pmatrix} \rho_+ \\ \rho_- \end{pmatrix} \in (L^\infty_{\text{Loc}})^2, s.t. \left| \begin{array}{l} \|\rho_+ - \rho_-\|_{L^\infty} \leq M \\ \rho \text{ is } 1-\text{periodic plus } L_0-\text{linear} \\ \max\limits_{k\in\{+,-\}} \|D\rho_k\|_{L^\infty} \leq B_0 \\ \max\limits_{k\in\{+,-\}} \|(\rho_k)_t\|_{L^\infty} \leq B_0(2M + \|L\|_{L^\infty(0,T)}) \end{array} \right. \right\},$$

where L_0 is defined in $(H2)$, $B_0 = \max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ and $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$. For $\rho \in U_T$, we set

$$a[\rho](t) = \int_0^1 \rho_+(x,t) - \rho_-(x,t)dx + L(t). \tag{4.34}$$

We see that for any $\rho \in U_T$, $a[\rho]$ satisfies $(H1)$ with $\|a[\rho]\|_{L^\infty(0,T)} \leq M + \|L\|_{L^\infty(0,T)}$.

For $\rho \in U_T$, we then define $v = G(\rho) = (G_+(\rho), G_-(\rho))$ as the unique viscosity solution for $k = 1, 2$ (see Theorem 4.8) of

$$\begin{cases} (v_k)_t = -k\left(v_+ - v_- + a[\rho](t)\right)|Dv_k| & \text{on} \quad (0,T)\times\mathbb{R}, \\ v_k(\cdot,0) = \rho_k^0 & \text{on} \quad \mathbb{R}. \end{cases} \tag{4.35}$$

We will show that $G : U_T \to U_T$ is a strict contraction for T small enough. First, we will prove that G is well defined. By Theorem 4.8, we know that v is $1-$ periodic plus L_0-linear. Moreover, we have

$$\max_{k\in\{+,-\}} \|Dv_k\|_{L^\infty(\mathbb{R}\times(0,T))} \leq B_0,$$

$$\max_{k\in\{+,-\}} \|(v_\pm)_t\|_{L^\infty(\mathbb{R}\times(0,T))} \leq B_0(M + \|a\|_{L^\infty(0,T)}) \leq B_0(2M + \|L\|_{L^\infty(0,T)})$$

and

$$\|v_+ - v_-\|_{L^\infty(\mathbb{R}\times(0,T))} \leq M$$

and so $v \in U_T$.

It thus remains to show that G is a contraction. For $v^i = G(\rho^i)$, according to *Lemma* 4.9 and *Lemma* 4.10, we have

$$\begin{aligned} \|v^2 - v^1\|_{L^\infty(\mathbb{R}\times(0,T))} &= \sup_{\{k\in\{+,-\}\}} \|v_k^2 - v_k^1\|_{L^\infty} \leq B_0 T\|a[\rho^2] - a[\rho^1]\|_{L^\infty(0,T)} \\ &\leq 2B_0T\|\rho^1 - \rho^2\|_{L^\infty(\mathbb{R}\times(0,T))} \leq \frac{1}{2}\|\rho^1 - \rho^2\|_{L^\infty(\mathbb{R}\times(0,T))} \end{aligned}$$

for $T \leq T^* = \dfrac{1}{4B_0}$. And so G is a contraction on U_T which is a closed set. So, there exists a unique viscosity solution of (1.1)-(1.2) in U_T on $(0, T^*)$ where $T^* = \dfrac{1}{4B_0}$. By iterating this process, one can construct a solution for all $T > 0$. Indeed, T^* depends only on B_0 which does not change with time. □

Proposition 4.11 (Estimate for the non-local solution) *Let $T \geq 0$. The solution ρ of (1.1)-(1.2) satisfies*

$$\|\rho_+ - \rho_-\|_{L^\infty(\mathbb{R}\times(0,T))} \leq M$$

where $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$.

The proof is the same of the one of the local case, see Theorem 4.8 (iv).

5 Numerical scheme

5.1 Approximation of the local system

In this subsection, we propose a finite difference scheme for the local system (4.19). Given a discrete velocity a^Δ, we consider the discrete solution v that approximates the solution of (4.19), given by the following explicit scheme for all $k \in \{+,-\}$

$$v_{k,i}^0 = \tilde{\rho}_k^0(x_i), \tag{5.36}$$

$$v_{k,i}^{n+1} = v_{k,i}^n + \Delta t \left(C_k^{\Delta,\mathrm{Loc}}[v](x_i,t_n) \right) \begin{cases} E^+\left(D^+ v_{k,i}^n, D^- v_{k,i}^n\right) & \text{if } C_k^{\Delta,\mathrm{Loc}}[v](x_i,t_n) \geq 0 \\ E^-\left(D^+ v_{k,i}^n, D^- v_{k,i}^n\right) & \text{if not} \end{cases} \tag{5.37}$$

where $\tilde{\rho}^0(x_i)$ are defined in (1.3), $E^\pm$ are the approximation of the Euclidean norm proposed by Osher and Sethian [116] defined in (1.6) (we also can use the one proposed by Rouy, Tourin [121]), $D^+ v_k^n$, $D^- v_k^n$ are the discrete gradient defined in (1.7) and

$$C_k^{\Delta,\mathrm{Loc}}[w](x_i,t_n) = -k(w_+(x_i,t_n) - w_-(x_i,t_n) + a^\Delta(t_n)) \tag{5.38}$$

where a^Δ is an approximation of a satisfying

$$a^\Delta(t_n) = a(t_n). \tag{5.39}$$

In particular, the functions $E^\pm$ are Lipschitz continuous with respect to the discrete gradients, *i.e.*

$$\left|E^\pm(P,Q) - E^\pm(P',Q')\right| \leq \left(|P-P'| + |Q-Q'|\right). \tag{5.40}$$

They are consistent with the Euclidean norm

$$E^\pm(P,P) = |P| \tag{5.41}$$

and enjoy suitable monotonicity with respect to each variable

$$\frac{\partial E^+}{\partial P^+} \geq 0, \quad \frac{\partial E^+}{\partial P^-} \leq 0, \quad \frac{\partial E^-}{\partial P^+} \geq 0, \quad \frac{\partial E^-}{\partial P^-} \leq 0. \tag{5.42}$$

Denoting by S^k the operator on the right-hand side of (5.37), we can rewrite the scheme more compactly as

$$v_{k,i}^0 = \tilde{\rho}_k^0(x_i), \quad v_k^{n+1} = S^k v^n.$$

Finally, we also assume that the mesh satisfies the following CFL condition (cf Remark 5.2)

$$\Delta t \leq \frac{1}{2L_1}\Delta x \tag{5.43}$$

where

$$L_1 = \|a\|_{L^\infty(0,T)} + M + 2.$$

Theorem 5.1 (Crandall-Lions rate of convergence) *Let $T \leq 1$. Assume that $\Delta x + \Delta t \leq 1$. Assume that $a \in W^{1,\infty}(\mathbb{R} \times [0,T))$ and that the CFL condition (5.43) holds. Then there exists a constant $K > 0$ depending only on $\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$, $\|a\|_{W^{1,\infty}(0,T)}$ and $\max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ such that the error estimate between the continuous solution ρ of the system (4.19) and the discrete solution v of the finite difference scheme (5.36)-(5.37) is given by*

$$\max_{k\in\{+,-\}} \sup_{\Xi_T} |\rho_k(x_i,t_n) - v_{k,i}^n| \leq K\sqrt{T}\,(\Delta x + \Delta t)^{1/2} + \max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0(x_i) - v_{k,i}^0|$$

provided $K(\Delta x + \Delta t)^{\frac{1}{2}} + \max_{k\in\{+,-\}} \sup_{\Xi}(\rho_k^0(x_i) - v_{k,i}^0) \leq 1$.

Remark 5.2 (Monotony of the scheme) *Under the assumptions of Theorem 5.1, we have*

$$\begin{aligned}|v_{+,i}^n - v_{-,i}^n| \leq & |v_{+,i}^n - \rho_+(x_i,t_n)| + |\rho_+(x_i,t_n) - \rho_-(x_i,t_n)| + |\rho_-(x_i,t_n) - v_{-,i}^n| \\ \leq & 2 + M\end{aligned}$$

where we have used Theorem 4.8 (iv) for the second term. We then deduce that the discrete velocity is uniformly bounded :

$$C_k^{\Delta,\mathrm{Loc}}[v] \leq \|a\|_{L^\infty(0,T)} + M + 2 = L_1.$$

Then, one can show that the scheme is monotone in the following sense : let v and w be two discrete functions such that $v_i^n \leq w_i^n$; then

$$S^k(v^n)(x_i) \leq S^k(w^n)(x_i), \quad \text{for } k \in \{+,-\}.$$

For the proof of Theorem 5.1, we need the following *lemma* :

***Lemma* 5.3** *If v_i^n is the numerical solution of (5.36)-(5.37), then*

$$-Kt_n - \mu^0 \leq \rho^0(x_i) - v(x_i,t_n) \leq Kt_n + \mu^0 \tag{5.44}$$

where $K = 2(\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})} + \|a\|_{L^\infty(0,T)}) \max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ and

$$\mu^0 = \max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0(x_i) - v_{k,i}^0| \geq 0. \tag{5.45}$$

Proof of Lemma 5.3

To prove this, we set $w_\pm(x_i,t_n) = \rho_\pm^0(x_i) - Kt_n - \mu^0$ and we show that for K large enough w is a discrete sub-solution. Indeed, we have

$$\begin{aligned}
& w_{\pm,i}^{n+1} - (S^\pm w^n)_i \\
= & -K\Delta t - \Delta t C_\pm^{\Delta,\mathrm{Loc}}[\rho^0](x_i,t_n) E^{sgn(C_\pm^{\Delta,\mathrm{Loc}}[\rho^0](x_i,t_n))}(D^+\rho_\pm^0(x_i), D^-\rho_\pm^0(x_i)) \\
= & -\Delta t \left(K \mp (\rho_+^0(x_i) - \rho_-^0(x_i) + a^\Delta(t_n)) E^{sgn(C_\pm^{\Delta,\mathrm{Loc}}[\rho^0](x_i,t_n))}(D^+\rho_\pm^0(x_i), D^-\rho_\pm^0(x_i))\right) \\
\leq & -\Delta t \left(K - 2(\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})} + \|a\|_{L^\infty(0,T)}) \max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}\right)
\end{aligned}$$

where $C_k^{\Delta,\mathrm{Loc}}[w](x_i,t_n)$ is defined in (5.38) and $sgn(f)$ is the sign of f.

So, for every $K \geq 2(\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})} + \|a\|_{L^\infty(0,T)}) \max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$, w is a discrete sub-solution. Moreover

$$w_{\pm,i}^0(x_i) = \rho_\pm^0(x_i) - \mu^0 \leq v_\pm^0(x_i).$$

Using the monotony of the scheme, we deduce $w_i^n \leq v_i^n$ and so

$$\rho^0(x_i) - v_i^n \leq Kt_n + \mu_0.$$

The lower bound is proved similarly. □

We now give the proof of Theorem 5.1

Proof of Theorem 5.1

The proof is an adaptation for systems of the one of Crandall Lions [40], revisited by Alvarez *et al.* [4]. The proof splits into three steps. We denote throughout by K various constant depending only on $\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$, $\max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ and $\|a\|_{W^{1,\infty(0,T)}}$.

We first assume that

$$\rho^0(x_i) \geq v_i^0 \tag{5.46}$$

and we set

$$\mu^0 = \max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0(x_i) - v_{k,i}^0| \geq 0. \tag{5.47}$$

We set a few notations. We put

$$\mu = \max_{k\in\{+,-\}} \sup_{\Xi_T} (\rho_k(x_i,t_n) - v_{k,i}^n).$$

For every $0 < \alpha \leq 1$, $0 < \varepsilon \leq 1$ and $\sigma > 0$, we set

$$M_\sigma^{\alpha,\varepsilon} = \sup_{\mathbb{R}\times[0,T]\times\Xi_T\times\{+,-\}} \Psi_\sigma^{\alpha,\varepsilon}(x,t,x_i,t_n,k),$$

with

$$\Psi_\sigma^{\alpha,\varepsilon}(x,t,x_i,t_n,k) = \rho_k(x,t) - v_k(x_i,t_n) - \frac{|x-x_i|^2}{2\varepsilon} - \frac{|t-t_n|^2}{2\varepsilon} - \sigma t - \alpha|x|^2 - \alpha|x_i|^2.$$

We shall drop the super and subscripts on Ψ when no ambiguity arises as concerning the value of the parameter.

Since ρ^0 is Lipschitz continuous and $T \leq 1$, we have by (4.29)

$$|\rho_\pm(x,t)| \leq K(1+|x|). \tag{5.48}$$

Moreover by *Lemma* 5.3 we have

$$\begin{aligned}|v_\pm(x_i,t_n)| \leq& |v_\pm(x_i,t_n)-\rho^0_\pm(x_i)|+|\rho^0_\pm(x_i)|\\ \leq& Kt_n+K(1+|x_i|)\\ \leq& K(1+|x_i|).\end{aligned}$$

We then deduce that Ψ achieves its maximum at some point that we denote by $(x^*,t^*,x_i^*,t_n^*,k^*)$.

Step 1 : Estimates for the maximum point of Ψ

The maximum point of Ψ enjoys the following estimates

$$\alpha|x^*|+\alpha|x_i^*|\leq K, \tag{5.49}$$

and

$$|x^*-x_i^*|\leq K\varepsilon, \quad |t^*-t_n^*|\leq (K+2\sigma)\varepsilon. \tag{5.50}$$

Indeed, by inequality $\Psi(x^*,t^*,x_i^*,t_n^*,k^*)\geq\Psi(0,0,0,0,k^*)\geq 0$, we obtain

$$\begin{aligned}\alpha|x^*|^2+\alpha|x_i^*|^2 \leq& \rho_{k^*}(x^*,t^*)-v_{k^*}(x_i^*,t_n^*)\leq K(1+|x^*|+|x_i^*|)\\ \leq& K+\frac{K^2}{\alpha}+\frac{\alpha}{2}|x^*|^2+\frac{\alpha}{2}|x_i^*|^2.\end{aligned}$$

This implies (5.49), since $\alpha\leq 1$.

The first bound of (5.50) follows from the Lipschitz in space regularity of ρ (see Theorem 4.8 (*iii*)), from the inequality $\Psi(x^*,t^*,x_i^*,t_n^*,k^*)\geq\Psi(x_i^*,t^*,x_i^*,t_n^*,k^*)$ and from (5.49). Indeed, this implies

$$\begin{aligned}\frac{|x^*-x_i^*|^2}{2\varepsilon} \leq& \rho_{k^*}(x^*,t^*)-\rho_{k^*}(x_i^*,t^*)-\alpha|x^*|^2+\alpha|x_i^*|^2\\ \leq& K|x^*-x_i^*|+\alpha|x^*-x_i^*|(|x^*|+|x_i^*|)\leq K|x^*-x_i^*|.\end{aligned}$$

The second bound of (5.50) is obtained in the same way, using the inequality $\Psi(x^*,t^*,x_i^*,t_n^*,k^*)\geq\Psi(x^*,t_n^*,x_i^*,t_n^*,k^*)$ and Theorem 4.8 (*iii*).

Step 2 : A better estimate for the maximum point of Ψ

Inequality (5.49) can be strengthened to

$$\alpha|x^*|^2+\alpha|x_i^*|^2\leq K. \tag{5.51}$$

Indeed, using the Lipschitz regularity of ρ, the inequality $\Psi(x^*,t^*,x_i^*,t_n^*,k^*)\geq\Psi(0,0,0,0,k^*)$ and equations (4.29), (5.44) and (5.50), yields

$$\begin{aligned}\alpha|x^*|^2+\alpha|x_i^*|^2 \leq& \rho_{k^*}(x^*,t^*)-v_{k^*}(x_i^*,t_n^*)+\rho(x_i^*,0)-\rho(x_i^*,0)\\ \leq& K(|x^*-x_i^*|+t^*)+Kt_n^*+\mu^0\leq K.\end{aligned}$$

Step 3 : Upper bound of μ

We have the bound $\mu \leq K\sqrt{T}\,(\Delta x + \Delta t)^{\frac{1}{2}} + \mu^0$ if $\Delta x + \Delta t \leq \frac{1}{K^2}$.
First, we claim that for σ large enough, we have either $t^* = 0$ or $t_n^* = 0$. Suppose the contrary. Then the function $(x,t) \mapsto \Psi(x,t,x_i^*,t_n^*,k^*)$ achieves its maximum at a point of $\mathbb{R} \times (0,T]$. Using the fact that ρ is a sub-solution of the continuous problem, we obtain the inequality

$$\sigma + p_t^* \leq -k^*(\rho_+ - \rho_- + a(t^*))|p_x^* + 2\alpha x^*| \tag{5.52}$$

with $p_t^* = \frac{t^*-t_n^*}{\varepsilon}$, $p_x^* = \frac{x^*-x_i^*}{\varepsilon}$.
Since $t_n^* > 0$, we also have $\Psi(x^*,t^*,x_i^*,t_n^*,k^*) \geq \Psi(x^*,t^*,x_i,t_n^*-\Delta t,k^*)$. This implies

$$v_{k^*}(\cdot,t_n^*-\Delta t) \geq \varphi(\cdot,t_n^*-\Delta t) + v_{k^*}(x_i^*,t_n^*) - \varphi(x_i^*,t_n^*)$$

for $\varphi(x_i,t_n) = -\frac{|x^*-x_i|^2}{2\varepsilon} - \frac{|t^*-t_n|^2}{2\varepsilon} - \alpha|x_i|^2$. Using the fact that the scheme is monotone and commutes with the addition of constants, yields

$$\begin{aligned}
&v_{k^*}(x_i^*,t_n^*)\\
=&S^{k^*}\left(v_{k^*}(\cdot,t_n^*-\Delta t)\right)(x_i^*)\\
\geq&\varphi(x_i^*,t_n^*-\Delta t) + v_{k^*}(x_i^*,t_n^*) - \varphi(x_i^*,t_n^*)\\
&+\Delta t\left(c_{k^*}^{\Delta,\mathrm{Loc}}[v](x_i^*,t_n^*)\right)E^{l^*}\left(D^+\varphi(x_i^*,t_n^*-\Delta t),D^-\varphi(x_i^*,t_n^*-\Delta t)\right)
\end{aligned}$$

where $l^* = sgn\left(c_{k^*}^{\Delta,\mathrm{Loc}}[v](x_i^*,t_n^*)\right)$. We set

$$c[v] = -c_{k^*}^{\Delta,\mathrm{Loc}}[v](x_i^*,t_n^*), \quad c[\rho] = k^*\left(\rho_+(x^*,t^*) - \rho_-(x^*,t^*) + a(t^*)\right).$$

We then obtain the super-solution inequality :

$$\frac{\varphi(x_i^*,t_n^*) - \varphi(x_i^*,t_n^*-\Delta t)}{\Delta t} \geq -c[v]E^{l^*}(D^+\varphi(x_i^*,t_n^*-\Delta t),D^-\varphi(x_i^*,t_n^*-\Delta t)).$$

Straightforward computations of the discrete derivative of φ yield

$$p_t^* + \frac{\Delta t}{2\varepsilon} \geq -c[v]E^{l^*}\left(p_x^* - \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^*+\Delta x), p_x^* + \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^*-\Delta x)\right).$$

Subtracting the above inequality to (5.52), we deduce

$$\begin{aligned}
\sigma \leq&\frac{\Delta t}{2\varepsilon} - c[\rho]|p_x^* + 2\alpha x^*| + c[v]E^{l^*}\left(p_x^* - \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^*+\Delta x), p_x^* + \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^*-\Delta x)\right)\\
\leq&\frac{\Delta t}{2\varepsilon} - \left(c[\rho] - c[v]\right)|p_x^*| + \alpha K|x^*|\\
&+|c[v]|\left|E^{l^*}\left(p_x^* - \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^*+\Delta x), p_x^* + \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^*-\Delta x)\right) - E^{l^*}\left(p_x^*,p_x^*\right)\right|\\
\leq&\frac{\Delta t}{2\varepsilon} - \left(c[\rho] - c[v]\right)|p_x^*| + K\alpha|x^*| + K\frac{\Delta x}{\varepsilon} + 2\alpha K|x_i^*| + 2\alpha K\Delta x
\end{aligned}$$

where we have used, for the second line, the fact that

$$c[\rho] \leq M + 2B_0(M + \|a\|_{L^\infty(0,T)})T \leq K$$

with $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$ and $B_0 = \max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ (see Theorem 4.8). Now, since $\rho_{k^*}(x^*,t^*) - v_{k^*}(x_i^*,t_n^*) = \max_{k\in\{+,-\}} (\rho_k(x^*,t^*) - v_k(x_i^*,t_n^*)) \geq 0$, by *Lemma* 4.2, we obtain

$$\begin{aligned}
-(c[\rho] - c[v])|p_x^*| =& - k^* \left(\rho_+(x^*,t^*) - \rho_-(x^*,t^*) + a(t^*)\right)|p_x^*| \\
&+ k^* \left(v_+(x_i^*,t_n^*) - v_-(x_i^*,t_n^*) + a(t^*)\right)|p_x^*| \\
&+ k^* \left(a^\Delta(t_n^*) - a(t^*)\right)|p_x^*| \\
\leq& \left|a^\Delta(t_n^*) - a(t^*)\right| |p_x^*| \leq K|t_n^* - t^*||p_x^*|
\end{aligned}$$

where we have used (5.39). This implies

$$\begin{aligned}
\sigma \leq& \frac{\Delta t}{2\varepsilon} + K|t^* - t_n^*||p_x^*| + K\alpha|x^*| + K\frac{\Delta x}{\varepsilon} + 2\alpha K|x_i^*| + 2\alpha K\Delta x \\
\leq& K\frac{\Delta x + \Delta t}{\varepsilon} + K\alpha^{1/2} + K\varepsilon.
\end{aligned}$$

Putting

$$\sigma^* = \sigma^*(\Delta x + \Delta t, \varepsilon, \alpha) = K\frac{\Delta x + \Delta t}{\varepsilon} + K(\alpha^{1/2} + \varepsilon),$$

we therefore conclude that we must have $t^* = 0$ or $t_n^* = 0$ provided $\sigma \geq \sigma^*$. Whenever $t^* = 0$, we deduce from *Lemma* 5.3 and from (5.50) that

$$\begin{aligned}
M_\sigma^{\alpha,\varepsilon} =& \Psi(x^*, 0, x_i^*, t_n^*, k^*) \leq \rho_{k^*}^0(x^*) - v_{k^*}(x_i^*, t_n^*) \\
\leq& \rho_{k^*}^0(x^*) - \rho_{k^*}^0(x_i^*) + Kt_n^* + \mu^0 \\
\leq& K\left(|x^* - x_i^*| + t_n^*\right) + \mu^0 \leq K(1+\sigma)\varepsilon + \mu^0.
\end{aligned}$$

Similarly, whenever $t_n^* = 0$, we deduce from the Lipschitz regularity of ρ and from (5.50) that

$$\begin{aligned}
M_\sigma^{\alpha,\varepsilon} =& \Psi(x^*, t^*, x_i^*, 0, k^*) \leq \rho_{k^*}(x^*, t^*) - v_{k^*}(x_i^*, 0) \\
\leq& K\left(|x^* - x_i^*| + t^*\right) + \mu^0 \leq K(1+\sigma)\varepsilon + \mu^0.
\end{aligned}$$

To sum up, we have shown that

$$M_\sigma^{\alpha,\varepsilon} \leq K(1+\sigma)\varepsilon + \mu^0 \leq K\varepsilon + \mu^0$$

provided $\sigma^* = K\frac{\Delta x + \Delta t}{\varepsilon} + K(\alpha^{1/2} + \varepsilon) \leq \sigma \leq 1$.We then deduce that, for every (x_i, t_n) and for every k, we have

$$\begin{aligned}
&\rho_k(x_i,t_n) - v_k(x_i,t_n) - \left(K\frac{\Delta x + \Delta t}{\varepsilon} + K(\alpha^{1/2} + \varepsilon)\right)T - 2\alpha|x_i|^2 \leq M_\sigma^{\alpha,\varepsilon} \\
\leq& K\varepsilon + \mu^0.
\end{aligned}$$

Sending $\alpha \to 0$, taking the supremum over (x_i, t_n), the maximum over k and choosing $\varepsilon = T^{1/2}(\Delta x + \Delta t)^{1/2}$, we conclude that

$$\begin{aligned}\max_{k\in\{+,-\}} \sup_{\Xi_T}(\rho_k(x_i,t_n) - v^n_{k,i}) =& \mu \\ \leq& K(\Delta x + \Delta t)^{1/2}\sqrt{T} + \max_{k\in\{+,-\}} \sup_{\Xi}(\rho^0_k(x_i) - v^0_{k,i}),\end{aligned} \tag{5.53}$$

provided that Δx, Δt are small enough $T \leq 1$, $\mu_0 \leq 1$ and (5.46) is assumed.

In the general case, we consider $\overline{\rho} = \rho + \mu^1$ with $\mu^1 = \max_{k\in\{+,-\}} \sup_{\Xi}(v^0_{k,i} - \rho^0_k(x_i))$. We remark that $\overline{\rho}$ is solution of (4.19) and satisfies $\overline{\rho}^0(x_i) \geq v^0_i$. Then (5.53) is true with $\overline{\rho}$ in place of ρ, *i.e.*

$$\max_{k\in\{+,-\}} \sup_{\Xi_T}(\rho_k(x_i,t_n)+\mu^1-v^n_{k,i}) \leq K(\Delta x + \Delta t)^{1/2}\sqrt{T}+\max_{k\in\{+,-\}} \sup_{\Xi}(\rho^0_k(x_i)+\mu^1-v^0_{k,i}),$$

which still implies (5.53) with $\max_{k\in\{+,-\}} \sup_{\Xi} |\rho^0_k(x_i) - v^0_{k,i}|$. □

The lower bound for the error estimate is obtained by exchanging ρ and v. As the proof is similar to the above, we omit it.

5.2 Approximation of the non-local system

To solve numerically the non-local system (1.1)-(1.2), we use the finite difference scheme (1.3)-(1.4)-(1.5). We also assume the CFL condition (1.8). In particular, using Proposition 4.11, we deduce that the CFL condition (5.43) is satisfied uniformly for all a defined by (1.5) because

$$\|a[\rho]\|_{L^\infty(0,T)} \leq M + \|L\|_{L^\infty(0,T)}$$

and so $L_1 \leq L_2$.

Let $\bar{T} \geq 0$ which will be chosen later. To prove our convergence result, we mimic the continuous case and we rewrite the scheme (1.3)-(1.4)-(1.5) as a fixed point. Before proving Theorem 1.3 we need to introduce some notations and *lemmata*. Defining $X^{1,\Delta}_T = \mathbb{R}^{\{0,\dots,N_T\}}$ and $X^{2,\Delta}_T = (\mathbb{R}^2)^{\mathbb{Z}\times\{0,\dots,N_T\}}$, the set of discrete functions defined on $\{0,\dots,N_T\}$ and on the mesh Ξ_T respectively, we denote by $G^\Delta : X^{1,\Delta}_{\bar{T}} \to X^{2,\Delta}_{\bar{T}}$ the operator that gives the discrete solution v of the local Problem (5.37) for a given velocity $a^\Delta \in X^{1,\Delta}_{\bar{T}}$, *i.e.*

$$(G^\Delta_+(a^\Delta), G^\Delta_-(a^\Delta)) = G^\Delta(a^\Delta) = v.$$

In particular, the scheme (1.3)-(1.4)-(1.5) can be rewritten as a fixed point of $G^\Delta(a^\Delta[\cdot])$, *i.e.*

$$v = G^\Delta(a^\Delta[v])$$

with $a^\Delta[\cdot]$ defined in (1.5). We set, for all $T \leq \bar{T}$:

$$U_T^\Delta = \left\{ w \in X_T^{2,\Delta} : \left| \begin{array}{l} \sup\limits_{\Xi_T} |D_x^+ w_\pm| \leq B_0, \\ \sup\limits_{\Xi_T} |D_t^+ w_\pm| \leq 2B_0(2M + \|L\|_{L^\infty(0,T)} + 4), \\ \sup\limits_{\Xi_T} |w_+ - w_-| \leq M + 2 \end{array} \right. \right\}$$

and

$$V_T^\Delta = \left\{ a^\Delta \in X_T^{1,\Delta} : \Big| \sup_{\{0,\dots,N_T\Delta t\}} |a^\Delta| \leq M + \|L\|_{L^\infty} + 2 \right\}$$

where $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$. One can easily check that

$$\{(\rho)^\Delta \mid \rho \in U_T\} \subset U_T^\Delta$$

and

$$\left\{ (a)^\Delta : \Big| \ \|a\|_{L^\infty(0,T)} \leq M + \|L\|_{L^\infty(0,T)} \ \right\} \subset V_T^\Delta$$

where $(f)^\Delta$ is the restriction to Ξ_T of the continuous function f. We have the following *Lemma* :

***Lemma* 5.4** *Assume that (1.8) holds. Then for all $T \leq \bar{T}$, the following inclusion holds*

(i) $a^\Delta[U_T^\Delta] \subset V_T^\Delta$,

(ii) $G^\Delta(V_T^\Delta) \subset U_T^\Delta$.

Proof of *Lemma* 5.4

The proof of (i) is just a simple computation. We prove (ii).

Let $a^\Delta \in V_T^\Delta$ and $v = G^\Delta(a^\Delta)$. We set $w(x_i, t_n) = v(x_{i+1}, t_n) - \Delta x B_0$. Then w is still solution of the discrete scheme (5.37) and satisfies $w^0 \leq v^0$. Using the monotony of the scheme yields

$$\frac{v_\pm(x_{i+1}, t_n) - v_\pm(x_i, t_n)}{\Delta x} \leq B_0.$$

Using Theorem 4.8, we deduce

$$|v_+ - v_-| \leq M + 2. \tag{5.54}$$

For the estimate in time, we have, using (5.54),

$$\begin{aligned} \left| \frac{v_i^{n+1} - v_i^n}{\Delta t} \right| \leq & 2B_0 |C_k^{\Delta,\mathrm{Loc}}[v](x_i, t_n)| \\ \leq & 2B_0(M + 2 + \sup_{\{0,\dots,N_T\Delta t\}} |a^\Delta|) \\ \leq & 2B_0(2M + \|L\|_{L^\infty(0,T)} + 4). \end{aligned}$$

So $G^\Delta(V_T^\Delta) \subset U_T^\Delta$. This ends the proof of the *lemma.* $\square$

We now have to prove some consistency and stability results for the velocity a^Δ and for the operator G^Δ.

***Lemma* 5.5 (Consistency for the discrete velocity $a^\Delta[\cdot]$)** *There is a constant $K = 2B_0 + M$ such that, for every mesh Δ, for every $0 \le T \le \bar{T}$ and for $\rho \in U_T$, we have*

$$\sup_{\{0,\dots,N_T\Delta t\}} |\,(a[\rho])^\Delta - a^\Delta[(\rho)^\Delta]| \le K\Delta x$$

where $(\rho)^\Delta$ is the restriction to Ξ_T of the continuous function ρ and $a[\cdot]$ is defined in (4.34).

Proof of *Lemma* 5.5
We set $\tilde{\rho}(x,t) = \rho_+(x,t) - \rho_-(x,t)$. The following holds :

$$\begin{aligned}
\left|a[\rho](t_n) - a^\Delta[(\rho)^\Delta](t_n)\right| &= \left|\int_0^1 \tilde{\rho}(x,t_n)dx - \sum_{i=0}^{N_x-1} \Delta x \tilde{\rho}(x_i,t_n)\right| \\
&\le \sum_{i=0}^{N_x-1} \left|\int_{i\Delta x}^{(i+1)\Delta x} \tilde{\rho}(x,t_n)dx - \Delta x \tilde{\rho}(x_i,t_n)\right| \\
&\quad + \int_{N_x\Delta x}^1 \tilde{\rho}(x,t_n)dx \\
&\le \Delta x \sum_{i=0}^{N_x-1} \sup_{[i\Delta x,(i+1)\Delta x]} |\tilde{\rho}(\cdot,t_n) - \tilde{\rho}(x_i,t_n)| + M\Delta x \\
&\le \Delta x(2B_0 + M).
\end{aligned}$$

$\square$

We have the following *lemma* which proof is just a simple computation

***Lemma* 5.6 (Stability property of the velocity $a^\Delta[\cdot]$)** *For every mesh Δ, for every $0 \le T \le \bar{T}$ and every $v_1, v_2 \in U_T^\Delta$, the following holds*

$$\sup_{\{0,\dots,N_T\Delta t\}} |a^\Delta[v_2] - a^\Delta[v_1]| \le 2 \max_{k\in\{+,-\}} \sup_{\Xi_T} |v_2 - v_1|.$$

***Lemma* 5.7 (Stability property of the operator G^Δ)** *There is a constant $K = 2B_0$ so that, for every mesh Δ satisfying the uniform CFL condition (1.8), for all $0 \le T \le \bar{T}$ and all a_1^Δ, $a_2^\Delta \in V_T^\Delta$*

$$\max_{k\in\{+,-\}} \sup_{\Xi_T} |G_k^\Delta(a_2^\Delta) - G_k^\Delta(a_1^\Delta)| \le KT \sup_{\{0,\dots,N_T\Delta t\}} |a_2^\Delta - a_1^\Delta|.$$

Proof of *Lemma* 5.7
We set $v_i = G^\Delta\left(a_i^\Delta\right)$. Using the fact that

$$c_1 E^{sgn(c_1)} - c_2 E^{sgn(c_2)} \leq |c_1 - c_2| \max(E^+, E^-)$$

yields

$$\begin{aligned}
&v_{2,k}^{n+1} - v_{2,k}^n + k\Delta t\left(v_{2,+}^n - v_{2,-}^n + a_1^\Delta(t_n)\right) E^{sgn(v_{2,+}^n - v_{2,-}^n + a_1^\Delta(t_n))}(D^+ v_2^n, D^- v_2^n)\\
\leq &\Delta t |a_2^\Delta(t_n) - a_1^\Delta(t_n)| \max(E^+(D^+ v_2^n, D^- v_2^n), E^-(D^+ v_2^n, D^- v_2^n))\\
\leq &2B_0 \Delta t \sup_{\{0,\dots,N_T\Delta t\}} |a_1^\Delta - a_2^\Delta|.
\end{aligned}$$

Moreover $\tilde{v}_1(x_i, t_n) = v_1(x_i, t_n) + 2B_0 \sup_{\{0,\dots,N_T\Delta t\}} |a_1^\Delta - a_2^\Delta| t_n$ is solution of the same discrete equation. Since the scheme is monotone, one deduces that

$$\max_{k\in\{+,-\}} \sup_{\Xi_T} |G_k^\Delta(a_2^\Delta) - G_k^\Delta(a_1^\Delta)| \leq 2B_0 T \sup_{\{0,\dots,N_T\Delta t\}} |a_2^\Delta - a_1^\Delta|.$$

This achieves the proof. □

We now prove Theorem 1.3.

Proof of Theorem 1.3
We use the main idea of Alvarez *et al.* [4].
We first assume that $T \geq \bar{T}$ and we set, for every $l \geq 1$:

$$Q_l^\Delta = \Delta x \mathbb{Z} \times \{\Delta t N_l, \dots, \Delta t N_{l+1}\}$$

where N_l is the integer part of $\dfrac{l\bar{T}}{\Delta t}$. As in the continuous case, on each interval $(l\bar{T}, (l+1)\bar{T})$, we can iterate the process (since $\bar{T}$ depends only on B_0 which does not change with time) and construct, using a fix point method (denoting by G and G^Δ), ρ and v respectively solution of (1.1)-(1.2) and (1.3)-(1.4)-(1.5). We then have the inequality

$$\begin{aligned}
\max_{k\in\{+,-\}} \sup_{Q_l^\Delta} |\rho_k - v_k| &\leq \max_{k\in\{+,-\}} \sup_{Q_l^\Delta} |G_{k,l}(a[\rho]) - G_{k,l}^\Delta(a^\Delta[v])|\\
&\leq \max_{k\in\{+,-\}} \sup_{Q_l^\Delta} \left|G_{k,l}(a[\rho]) - G_{k,l}^\Delta\left((a[\rho])^\Delta\right)\right|\\
&\quad + \max_{k\in\{+,-\}} \sup_{Q_l^\Delta} \left|G_{k,l}^\Delta\left((a[\rho])^\Delta\right) - G_{k,l}^\Delta\left(a^\Delta[v]\right)\right|,
\end{aligned}$$

where the function $G_l^\Delta\left((a[\rho])^\Delta\right) = \left(G_{+,l}^\Delta\left((a[\rho])^\Delta\right), G_{-,l}^\Delta\left((a[\rho])^\Delta\right)\right)$ (resp. $G_l(a[\rho])$) is simply the discrete solution of (5.36) (resp. the continuous solution of (1.1)) with

the velocity $a[\rho]$ and initial condition v^{N_l} (resp. $\rho(\cdot, N_l\mathcal{D}t)$). >From Theorem 5.1, we then deduce

$$\max_{k\in\{+,-\}} \sup_{Q_l^\Delta} \left|G_{k,l}(a[\rho]) - G_{k,l}^\Delta\left((a[\rho])^\Delta\right)\right| \leq K\sqrt{\bar{T}\Delta x} + \max_{k\in\{+,-\}} \sup_{\Delta x\mathbb{Z}\times N_l\Delta t} |\rho_k - v_k|$$

$$\leq lK\sqrt{\bar{T}\Delta x} + \max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0|. \quad (5.55)$$

For the second term, we use *Lemmata* 5.5, 5.6 and 5.7 to obtain

$$\begin{aligned}
&\max_{k\in\{+,-\}} \sup_{Q_l^\Delta} \left|G_{k,l}^\Delta\left((a[\rho])^\Delta\right) - G_{k,l}^\Delta\left(a^\Delta[v]\right)\right| \\
\leq& K\bar{T} \sup_{\{N_l\Delta t,\ldots,N_{l+1}\Delta t\}} \left|(a[\rho])^\Delta - a^\Delta[v]\right| \\
\leq& K\bar{T} \sup_{\{N_l\Delta t,\ldots,N_{l+1\Delta t}\}} \left(\left|\,(a[\rho])^\Delta - a^\Delta\left[(\rho)^\Delta\right]\,\right| + \left|a^\Delta\left[(\rho)^\Delta\right] - a^\Delta[v]\right|\right) \\
\leq& K\bar{T}\left(\Delta x + \max_{k\in\{+,-\}} \sup_{Q_l^\Delta} |\rho_k - v_k|\right).
\end{aligned}$$

This implies, for $\bar{T}\Delta x \leq 1$ and $K\bar{T} < 1$,

$$\max_{k\in\{+,-\}} \sup_{Q_l^\Delta} |\rho_k - v_k| \leq \frac{lK}{1-K\bar{T}}\sqrt{\bar{T}\Delta x} + \left(\max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0|\right)\frac{1}{1-K\bar{T}}.$$

We now take $\bar{l} \geq 1$ such that

$$\bar{l}\bar{T} \leq T \leq (\bar{l}+1)\bar{T}.$$

Then the following holds :

$$\begin{aligned}
\max_{k\in\{+,-\}} \sup_{\Xi_T} |\rho_k - v_k| \leq& \frac{\bar{l}K}{1-K\bar{T}}\sqrt{\bar{T}\Delta x} + \left(\max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0|\right)\frac{1}{1-K\bar{T}} \\
\leq& KT\sqrt{\Delta x} + K\left(\max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0|\right), \text{ if } T \geq \bar{T}.
\end{aligned}$$

where we have used the fact that $\bar{T}$ depends only on B_0.

Notice that, in the case where $T \leq \bar{T}$, from Theorem 5.1, (5.55) is replaced by

$$\max_{k\in\{+,-\}} \sup_{\Xi_T} \left|G_k(a[\rho]) - G_k^\Delta\left((a[\rho])^\Delta\right)\right| \leq K\sqrt{T\Delta x} + \max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0|$$

and so we obtain

$$\max_{k\in\{+,-\}} \sup_{\Xi_T} |\rho_k - v_k| \leq K\sqrt{T\Delta x} + K\left(\max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0|\right), \text{ if } T \leq \bar{T}.$$

This ends the proof of the theorem. □

6 Numerical results

In this Section, we present some numerical simulations of the 1-D Groma-Balogh problem (1.1)-(1.2) discretized by the numerical scheme (1.3)-(1.4)-(1.5).

6.1 Numerical error estimate

Here, we show a numerical test in order to confirm our error estimate for local system. Let us fix $L(t) = 0$ even if it is not physicaly relevant, let us choose the following initial conditions : $\rho_+^0(x) = -|x - 1/2| + 1/2$, and $\rho_-^0(x) = -|2x - 1| + 1$ on $[0, 1[$ (and extend it by periodicity on $\mathbb{R}$).

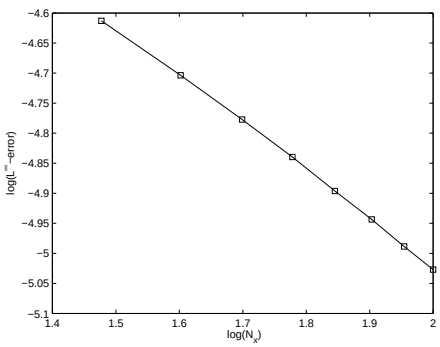

FIG. 3.3 – $\log(L^\infty - error)$ of $|u_{N_T} - u_{N_T-1}|$ versus $\log(N_x)$ at $T = \frac{1}{2}$

Figure (3.3) show the behaviour of the L^∞-error versus the discretization parameter Δx. The regression slope is close to 0.7 and the ideal regression is $\frac{1}{2}$. Hence, the behaviour of this errors confirms that our error seems optimal.

6.2 Dislocation density dynamics

In this paragraph, we are interested by the evolution of dislocations densities for the 1-D Groma-Balogh model (1.1)-(1.2) under the uniformly applied shear stress $L(t) = 3t$.

In this simulations, we choose an example of concentrated dislocations densities, *i.e.* where dislocations densities are initially periodic, and equal to zero on some sub-intervals of $[0, 1[$ (see Figure 3.4).

This initial condition means that there exists some regions without dislocations, and others with concentrated dislocations.

Intuitively, dislocations are intended to be uniformly distributed in the whole crystal as shown in (Figure 3.6) where finally a uniform distribution in all the crystal is observed, *i.e.* the density of dislocations becomes a constant.

We remark that when $L(t)$ is non-stationary, our system behaves as a diffusion equation (see [22] for further details). But evidently when $L(t) = 0$, with the same initial condition, the system does not evolve.

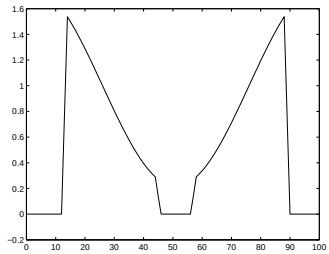

FIG. 3.4 – dislocations density $(D\rho_+^0(.) = D\rho_-^0(.))$

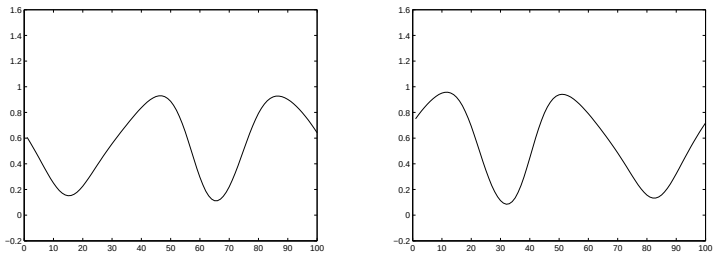

FIG. 3.5 – On the left : density $(D\rho_+(., \frac{1}{2}))$; on the right : dislocations density $(D\rho_-(., \frac{1}{2}))$

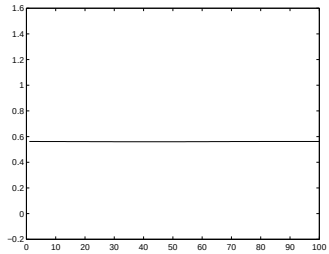

FIG. 3.6 – dislocations density $(D\rho_+(.,3) = D\rho_-(.,3))$

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Chapitre 4

Existence globale de solutions continues pour un système hyperbolique diagonalisable avec des données initiales grandes et monotones

Ce chapitre est un travail en collaboration avec R. Monneau.

Dans ce travail, nous nous intéressons à l'étude des systèmes hyperboliques diagonalisables en dimension 1. En se basant sur une nouvelle estimation sur l'entropie du gradient des solutions, nous prouvons l'existence globale d'une solution continue, pour des données initiales grandes et croissantes. De plus, nous montrons dans un cas particulier quelques résultats d'unicité. Nous remarquons également que ces résultats couvrent le cas des systèmes qui sont hyperboliques mais pas strictement hyperboliques. Physiquement, ce genre de systèmes hyperboliques diagonalisables apparaît naturellement dans la modélisation de la dynamique des densités de dislocations.

Global continuous solutions to diagonalizable hyperbolic systems with large and monotone data

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Abstract

In this paper, we study diagonalizable hyperbolic systems in one space dimension. Based on a new gradient entropy estimate, we prove the global existence of a continuous solution, for large and nondecreasing initial data. Moreover, we show in particular cases some uniqueness results. We also remark that these results cover the case of systems which are hyperbolic but not strictly hyperbolic. Physically, this kind of diagonalizable hyperbolic systems appears naturally in the modelling of the dynamics of dislocation densities.

AMS Classification : 35L45, 35Q35, 35Q72, 74H25.
Key words : Global existence, system of Burgers equations, system of nonlinear transport equations, nonlinear hyperbolic system, dynamics of dislocation densities.

1 Introduction and main result

1.1 Setting of the problem

In this paper we are interested in continuous solutions to hyperbolic systems in dimension one. Our work will focus on solution $u(t,x) = (u^i(t,x))_{i=1,\dots,M}$, where M is an integer, of hyperbolic systems which are diagonal, i.e.

$$\partial_t u^i + a^i(u)\partial_x u^i = 0 \quad \text{on} \quad (0,T)\times\mathbb{R} \quad \text{and for} \quad i = 1,\dots,M, \tag{P}$$

with the initial data :

$$u^i(0,x) = u_0^i(x), \qquad x \in \mathbb{R}, \text{ for } i = 1, \dots, M. \tag{ID}$$

For real numbers $\alpha^i \leq \beta^i$, let us consider the box

$$U = \Pi_{i=1}^M [\alpha^i, \beta^i]. \tag{1.1}$$

We consider a given function $a = (a^i)_{i=1,\dots,M} : U \to \mathbb{R}^M$, which satisfies the following regularity assumption :

$$(H1) \quad \begin{cases} \text{the function } a \in C^\infty(U), \\ \\ \text{there exists} \quad M_0 > 0 \quad \text{such that for} \quad i = 1, ..., M, \\ |a^i(u)| \leq M_0 \quad \text{for all} \quad u \in U, \\ \\ \text{there exists} \quad M_1 > 0 \quad \text{such that for} \quad i = 1, ..., M, \\ |a^i(v) - a^i(u)| \leq M_1 |v - u| \quad \text{for all} \quad v, u \in U. \end{cases}$$

We assume, for all $u \in \mathbb{R}^M$, that the matrix

$$(a^i_{,j}(u))_{i,j=1,\dots,M}, \text{ where } a^i_{,j} = \frac{\partial}{\partial u^j} a^i,$$

is non-negative in the positive cone, namely

$$(H2) \quad \left| \begin{array}{l} \text{for all} \quad u \in U, \quad \text{we have} \\ \\ \displaystyle\sum_{i,j=1,\dots,M} \xi_i \xi_j a^i_{,j}(u) \geq 0 \quad \text{for every} \quad \xi = (\xi_1, ..., \xi_M) \in [0, +\infty)^M. \end{array} \right.$$

In (ID), each component u_0^i of the initial data $u_0 = (u_0^1, \cdots, u_0^M)$ is assumed satisfy the following property :

$$(H3) \quad \left\{ \begin{array}{l} u_0^i \in L^\infty(\mathbb{R}), \\ \\ u_0^i \text{ is nondecreasing,} \\ \\ \partial_x u_0^i \in L \log L(\mathbb{R}), \end{array} \right| \quad \text{for } i = 1, \cdots, M,$$

where $L \log L(\mathbb{R})$ is the following Zygmund space :

$$L \log L(\mathbb{R}) = \left\{ f \in L^1(\mathbb{R}) \text{ such that } \int_{\mathbb{R}} |f| \ln(1 + |f|) < +\infty \right\}.$$

This space is equipped by the following norm :

$$\|f\|_{L\log L(\mathbb{R})} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}} \frac{|f|}{\lambda} \ln\left(1 + \frac{|f|}{\lambda}\right) \leq 1\right\},$$

This norm is due to Luxemburg (see Adams [2, (13), Page 234]).

Our purpose is to show the existence of a continuous solution, such that $u^i(t,\cdot)$ satisfies $(H3)$ for all time.

1.2 Main result

It is well-known that for the classical Burgers equation, the solution stays continuous when the initial data is Lipschitz-continuous and non-decreasing. We want somehow to generalize this result to the case of diagonal hyperbolic systems.

Theorem 1.1 ***(Global existence of a nondecreasing solution)***
Assume $(H1)$, $(H2)$ and $(H3)$. Then, for all $T > 0$, we have :

i) **Existence of a weak solution :**
There exists a function u solution of (P)-(ID) (in the distributional sense), where

$$u \in [L^\infty((0,T)\times\mathbb{R})]^M \cap [C([0,T); L\log L(\mathbb{R}))]^M \text{ and } \partial_x u \in [L^\infty((0,T); L\log L(\mathbb{R}))]^M,$$

such that for a.e $t \in [0,T)$ the function $u(t,\cdot)$ is nondecreasing in x and satisfies the following L^∞ estimate :

$$\|u^i(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \|u_0^i\|_{L^\infty(\mathbb{R})}, \qquad \text{for } i = 1,\dots,M, \tag{1.2}$$

and the gradient entropy estimate :

$$\int_{\mathbb{R}} \sum_{i=1,\dots,M} f\left(\partial_x u^i(t,x)\right) dx + \int_0^t \int_{\mathbb{R}} \sum_{i,j=1,\dots,M} a^i_{,j}(u)\partial_x u^i(s,x)\partial_x u^j(s,x)\, dx\, ds \leq C_1, \tag{1.3}$$

where

$$f(x) = \begin{cases} x\ln(x) + \frac{1}{e} & \text{if } x \geq 1/e, \\ 0 & \text{if } 0 \leq x \leq 1/e, \end{cases} \tag{1.4}$$

and $C_1(T, M, M_1, \|u_0\|_{[L^\infty(\mathbb{R})]^M}, \|\partial_x u_0\|_{[L\log L(\mathbb{R})]^M})$.

ii) **Continuity of the solution :**
The solution u constructed in (i) belongs to $C([0,T)\times\mathbb{R})$ and there exists a modulus of continuity $\omega(\delta,h)$, such that for all $(t,x) \in (0,T)\times\mathbb{R}$ and all $\delta, h \geq 0$, we have :

$$|u(t+\delta, x+h) - u(t,x)| \leq C_2\, \omega(\delta, h) \;\; \textit{with} \;\; \omega(\delta, h) = \frac{1}{\ln(\frac{1}{\delta}+1)} + \frac{1}{\ln(\frac{1}{h}+1)}. \quad (1.5)$$

where $C_2(T, M_1, M_0, \|u_0\|_{[L^\infty(\mathbb{R})]^M}, \|\partial_x u_0\|_{[L\log L(\mathbb{R})]^M})$.

Remark 1.2
Here, we can easily extend the solution u of (P)-(ID), given by Theorem 1.1, on the time interval $[0, +\infty)$.

Our method is based on the following simple remark : if the initial data satisfies $(H3)$ then the solution satisfies $(H3)$ for all t. What seems new is the gradient entropy inequality. The prove of Theorem 1.1 is rather standard. First we regularize the initial data and the system with the addition of a viscosity term, then we show that this regularized system admits a classical solution for short time. We prove the bounds (1.2) and the fundamental gradient entropy inequality (1.3) which allow to get a solution for all time. Finally, these *a priori* estimates ensure enough compactness to pass to the limit when the regularization varnishes and to get the existence of a solution.

Remark 1.3
To guarantee the $L\log L$ bound on the gradient of the solutions. We assumed in (H2) a sign on the left hand side of gradient entropy inequality (1.3).

In the case of 2×2 strictly hyperbolic systems, which corresponds in (P) to the case of $a^1(u^1,u^2) < a^2(u^1,u^2)$. Lax [98] proved the existence of smooth solution of (P)-(ID). This result was also proven by Serre [123, Vol II] in the case of $M\times M$ rich hyperbolic systems (see also Subsection 1.4 for more related references). Their result is limited to the case of strictly hyperbolic systems, here in Theorem 1.1, we treated the case of systems which are hyperbolic but not strictly hyperbolic. See the following Remark for a quite detailed example.

Remark 1.4 (Crossing eigenvalues)
Condition (1.9) on the eigenvalues is required in our framework (Theorem 1.1). Here is a simple example of a 2×2 hyperbolic but not strictly hyperbolic system. We consider solution $u = (u^1, u^2)$ of

$$\left\{\begin{array}{l} \partial_t u^1 + cos(u^2)\partial_x u^1 = 0, \\ \\ \partial_t u^2 + u^1 sin(u^2)\partial_x u^2 = 0, \end{array}\right| \quad on\ (0,T)\times\mathbb{R}. \quad (1.6)$$

Assume :

i) $u^1(-\infty) = 0$, $u^1(+\infty) = 1$ *and* $\partial_x u^1 \geq 0$,

ii) $u^2(-\infty) = -\frac{\pi}{2}$, $u^2(+\infty) = \frac{\pi}{2}$ *and* $\partial_x u^2 \geq 0$.

Here the eigenvalues $\lambda_1(u^1, u^2) = cos(u^2)$ *and* $\lambda_2(u^1, u^2) = u^1 sin(u^2)$ *cross each other at the initial time (and indeed for any time). Nevertheless for* $a^1(u^1, u^2) = cos(u^2)$ *and* $a^2(u^1, u^2) = u^1 sin(u^2)$, *we can compute*

$$(a^i_{,j}(u^1, u^2))_{i,j=1,2} = \begin{pmatrix} 0 & -sin(u^2) \\ sin(u^2) & u^1 cos(u^2) \end{pmatrix},$$

which satisfies $(H2)$ *(under assumptions (i) and (ii)). Therefor Theorem 1.1 gives the existence of a solution to (1.6) with (i) and (ii).*

Based on the same type of gradient entropy inequality (1.3), it was proved in Cannone et al. [25] the existence of a solution in the distributional sense for a two-dimensional system of two transport equations, where the velocity vector field is non-local.

The uniqueness of the solution is strongly related to the existence of regular (Lipschitz) solutions (see Theorem 7.7). Let us remark that equation (P)-(ID) does not create shocks because the solution (given in Theorem 1.1) is continuous. In this situation, it seems very natural to expect the uniqueness of the solution. Indeed the notion of entropy solution (in particular designed to deal with the discontinuities of weak solutions) does not seem so helpful in this context. Nevertheless the uniqueness of the solution is an open problem in general (even for such a simple system).

We ask the following **Open question :**
Is there uniqueness of the solution given in Theorem 1.1 ?

Now we give the following existence and uniqueness result in $[W^{1,\infty}([0,T) \times \mathbb{R})]^M$, in a special case to simplify the presentation. More precisely we assume

$$(H1') \quad a^i(u) = \sum_{j=1,\dots,M} A_{ij} u^j \text{ for } i = 1, \dots, M \text{ and for all } u \in U,$$

$$(H2') \quad \sum_{i,j=1,\dots,M} A_{ij}\xi_i\xi_j \geq 0 \quad \text{for every} \quad \xi = (\xi_1, ..., \xi_M) \in [0, +\infty)^M.$$

Theorem 1.5 (Existence and uniqueness of $W^{1,\infty}$ solution for a particular $A = (A_{ij})_{i,j=i=1,\dots,M}$)

Assume $(H1')$. *For* $T > 0$ *and all nondecreasing initial data* $u_0 \in [W^{1,\infty}(\mathbb{R})]^M$, *the system (P)-(ID) admits a unique solution* $u \in [W^{1,\infty}([0,T) \times \mathbb{R})]^M$, *in the following cases :*

i) $M \geq 2$ *and* $A_{ij} \geq 0$, *for all* $j \geq i$.
ii) $M \geq 2$ *and* $A_{ij} \leq 0$, *for all* $i \neq j$ *and* $(H2')$. *And then for all* $(t,x) \in [0,T) \times \mathbb{R}$ *we have*

$$\sum_{i=1,\dots,M} \partial_x u^i(t,x) \leq \sup_{y\in\mathbb{R}} \sum_{i=1,\dots,M} \partial_x u_0^i(y). \tag{1.7}$$

Remark 1.6 (Case of $M = 2$)
In particular for $M = 2$, *if* $(H1')$, $(H2')$ *and* $(H3)$ *satisfied then we have, by Theorem 1.5 the existence and uniqueness of a solution in* $[W^{1,\infty}([0,T) \times \mathbb{R})]^2$ *of (P)-(ID).*

In these particular cases of the matrix A, we can prove that $\partial_x u^i$ for $i = 1, \dots, M$, are bounded on $[0,T) \times \mathbb{R}$. Thanks to this better estimates on $\partial_x u^i$, and then on the velocity vector field Au, we prove here the uniqueness of the solution.

In the case of the matrix $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, it was proved in El Hajj, Forcadel [48], the existence and uniqueness of a Lipschitz viscosity solution, and in A. El Hajj [47], the existence and uniqueness of a strong solution in $W^{1,2}_{loc}([0,T) \times \mathbb{R})$.

1.3 Application to diagonalizable systems

Let us first consider a smooth function $u = (u^1, \dots, u^M)$, solution of the following non-conservative hyperbolic system :

$$\begin{cases} \partial_t u(t,x) + F(u)\partial_x u(t,x) = 0, & u(t,x) \in U,\ x \in \mathbb{R},\ t \in (0,T), \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{1.8}$$

where the space of states U is now an open subset of $\mathbb{R}^M$, and for each u, $F(u)$ is a $M \times M$-matrix and the map F is of class $C^1(U)$. We assume that $F(u)$ has M real eigenvalues $\lambda_1(u), \dots, \lambda_M(u)$, and we suppose that we can select bases of right and left eigenvectors $r_i(u)$, $l_i(u)$ normalized so that

$$|r_i| \equiv 1 \quad \text{and} \quad l_i \cdot r_j = \delta_{ij}$$

Remark 1.7 (Riemann invariant)
Recall that locally a necessary and sufficient condition to write

$$l_i(u) = \nabla_u \varphi_i(u),$$

is the Frobenius condition $l_i \wedge dl_i = 0$. In that case the function $\varphi_i(u)$ is solution of the following equation

$$(\varphi_i(u))_t + \lambda_i(u)(\varphi_i(u))_x = 0.$$

We recall that then $\varphi_i(u)$ is called a i-Riemann invariant (see Sevennec [124] and Serre [123, Vol II])). If this is true for any i, we say that the system (1.8) is diagonalizable.

Our theory is naturally applicable to this more general class of systems.

1.4 A brief review of some related literature

Now we recall some well known results for system (1.8).
For a scalar conservation law, this corresponds in (1.8) to the case $M = 1$ and $F(u) = h'(u)$ is the derivative of some flux function h, the global existence and uniqueness of BV solution established by Oleinik [113] in one space dimension. The famous paper of Kruzhkov [93] covers the more general class of L^∞ solutions, in several space dimension. For another alternative approach based on the notion of entropy process solutions, see Eymard et al. [52], see also the kinetic formulation P. L. Lions et al. [106].

We now recall some well-known results for a class of 2×2 strictly hyperbolic systems n one space dimension. Here i.e $F(u)$ has two real, distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u).$$

Lax [98] proved the existence and uniqueness of nondecreasing and smooth solutions of the 2×2 strictly hyperbolic systems. Also in case of 2×2 strictly hyperbolic systems DiPerna [43, 44] showed the global existence of a L^∞ solution. The proof of DiPerna relies on a compensated compactness argument, based on the representation of the weak limit in terms of Young measures, which must reduce to a Dirac mass due to the presence of a large family of entropies. This results is based on the idea of Tartar [127]. D. Serre [122] has studied the case of (2×2) Temple systems for which one has global existence for data with bounded variation.

For general $M \times M$ strictly hyperbolic systems ; i. e. where $F(u)$ has M real, distinct eigenvalues

$$\lambda_1(u) < \cdots < \lambda_M(u), \tag{1.9}$$

Bianchini and Bressan proved in [17] a striking global existence and uniqueness result of BV solutions to system (1.8), assuming that the initial data has small total variation. Their existence result is a generalization of Glimm result [64], proved in the conservation case ; i.e. $F(u) = Dh(u)$ is the Jacobin of some flux function h and generalized by LeFloch and Liu [99, 100] in the non-conservative case.

We can also mention that, our system (P) is related to other similar models, such as scalar transport equations based on vector fields with low regularity. Such equations were for instance studied by Diperna and Lions in [45]. They have proved the existence (and uniqueness) of a solution (in the renormalized sense), for given vector fields in $L^1((0,+\infty); W^{1,1}_{loc}(\mathbb{R}^N))$ whose divergence is in $L^1((0,+\infty); L^\infty(\mathbb{R}^N))$. This study was generalized by Ambrosio [8], who considered vector fields in $L^1((0,+\infty); BV_{loc}(\mathbb{R}^N))$ with bounded divergence. In the present paper, we work in dimension $N = 1$ and prove the existence (and some uniqueness results) of solutions of the system (P)-(ID) with a velocity vector field $a^i(u)$, $i = 1, \ldots, M$. Here, in Theorem 1.1, the divergence of our vector field is only in $L^\infty((0,+\infty), L\log L(\mathbb{R}))$. In this case we proved the existence result thanks to the gradient entropy estimate (1.3), which gives a better estimate on the solution. However, in Theorem 1.5, the divergence of our vector field is bounded, which allows us to get a uniqueness result for the non-linear system (P).

We also refer to Ishii, Koike [83] and Ishii [82], who showed existence and uniqueness of viscosity solutions for Hamilton-Jacobi systems of the form :

$$\begin{cases} \partial_t u^i + H_i(u, Du^i) = 0 \quad \text{with} \;\; u = (u^i)_i \in \mathbb{R}^M, \;\text{for} \;\; x \in \mathbb{R}^N, \; t \in (0,T), \\ \\ u^i(x,0) = u^i_0(x) \qquad\qquad x \in \mathbb{R}, \end{cases} \tag{1.10}$$

where the Hamiltonian H_i is quasi-monotone in u (see Ishii, Koike [83, Th.4.7]). This does not cover our study since our Hamiltonian is not necessarily quasi-monotone.

For hyperbolic and symmetric systems, Gårding has proved in [59] a local existence and uniqueness result in $C([0,T); H^s(\mathbb{R}^N)) \cap C^1([0,T); H^{s-1}(\mathbb{R}^N))$, with $s > \frac{N}{2} + 1$ (see also Serre [123, Vol I, Th 3.6.1]), this result being only local in time, even in dimension $N = 1$.

1.5 Miscellaneous extensions to explore in a futur work

1. In Theorem 1.1 we have considered the study of a particular system only to simplify the presentation. This result could be generalized to the following system

$$\partial_t u^i + a^i(u,x,t)\partial_x u^i = h^i(u,x,t) \quad \text{on} \quad (0,T)\times\mathbb{R} \quad \text{and for} \quad i=1,...,M, \qquad \text{(P')}$$

with suitable conditions on a^i and h^i.

2. If we consider the case where the system (P) is strictly hyperbolic. Based in the result of Bianchini, Bressan [17], we could also prove the uniqueness of the solution, whose existence is given by Theorem 1.1.
3. We could also extend Theorem 1.5 to system (P'), where we replace (i) and (ii) by the following condition

i') For $M \geq 2$, $a^i_j(u,x,t) \geq 0$ for $j \geq i$ and for all $(u,x,t) \in U \times \mathbb{R} \times [0,T)$.
ii') For $M \geq 2$,

$$a^i_{,j}(u,x,t) \leq 0 \quad \text{for all} \quad (u,x,t) \in U \times \mathbb{R} \times [0,+\infty), \quad \text{for all} \quad i \neq j,$$

and we assume that for any $v_i \in \mathbb{R}^M, x_i \in \mathbb{R}$, the matrix

$$b_{ij}(t) = a^i_{,j}(v_i, x_i, t)$$

satisfies for all $t \geq 0$

$$(H2'') \qquad \sum_{i,j=1,...,M} b_{ij}(t)\xi_i\xi_j \geq 0 \quad \text{for all} \quad \xi = (\xi_1,...,\xi_M) \in [0,+\infty)^M.$$

4. We could also prove the uniqueness result in case of $W^{1,\infty}$ solution among weak solution. (and in particular any weak solution is a viscosity solution in the sense of Crandall-Lions [38, 39]).

5. We could propose a numerical scheme and try to prove its convergence.

6. Applications to other equations : Euler, p-systems.

1.6 Organization of the paper

This paper is organized as follows : in the Section 2, we approximate the system (P) and the initial conditions. Then we prove a local in time existence for this approximated system. In Section 3, we prove the global in time existence for the approximated system. In the Section 4, we prove that the obtained solutions are regular and non-decreasing in x for all $t \in (0,T)$. In Section 5, we prove the gradient entropy inequality and some other ε-uniform *a priori* estimates. In Section 6, we prove the main result (Theorem 1.1) passing to the limit as ε goes to 0 and using

some compactness properties inherited from our entropy gradient inequality and the *a priori* estimates. In Section 7 we prove some uniqueness results in particular cases (Theorem 1.5). An application to the dynamics of dislocation densities given in Section 8. Finally, in the Appendix, we recall the proof of uniqueness of Lipschitz solution to system (P).

2 Local existence of an approximated system

The system (P) can be written as :

$$\partial_t u + a(u) \diamond \partial_x u = 0, \tag{2.11}$$

where $u := (u^i)_{1,\dots,M}$, $a(u) = (a^i(u))_{1,\dots,M}$ and $U \diamond V$ is the "component by component product" of the two vectors $U, V \in \mathbb{R}^M$. This is the vector in $\mathbb{R}^M$ whose coordinates are given by $(U \diamond V)_i := U_i V_i$:

$$\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_M \end{bmatrix} \diamond \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_M \end{bmatrix} = \begin{bmatrix} U_1 V_1 \\ U_2 V_2 \\ \vdots \\ U_M V_M \end{bmatrix}.$$

Now, we consider the system (2.11), modified by the term $\varepsilon \partial_{xx} u$, where $\partial_{xx} = \dfrac{\partial^2}{\partial x^2}$, and for smoothed data. This modification brings us to study, for all $0 < \varepsilon \leq 1$, the following system :

$$\partial_t u^\varepsilon - \varepsilon \partial_{xx} u^\varepsilon = -a(u^\varepsilon) \diamond \partial_x u^\varepsilon, \tag{P_ε}$$

with the smooth initial data :

$$u^\varepsilon(x, 0) = u_0^\varepsilon(x), \quad \text{with } u_0^\varepsilon(x) := u_0 * \eta_\varepsilon(x), \tag{ID_ε}$$

where η_ε is a mollifier verify, $\eta_\varepsilon(\cdot) = \frac{1}{\varepsilon}\eta(\frac{\cdot}{\varepsilon})$, such that $\eta \in C_c^\infty(\mathbb{R})$ is a non-negative function and $\int_{\mathbb{R}} \eta = 1$.

Remark 2.1
By classical properties of the mollifier $(\eta_\varepsilon)_\varepsilon$ and the fact that $u_0^\varepsilon \in [L^\infty(\mathbb{R})]^M$, then $u_0^\varepsilon \in [C^\infty(\mathbb{R})]^M \cap [W^{m,\infty}(\mathbb{R})]^M$ for all $m \in \mathbb{N}$.

The global existence of smooth solution of the system (P_ε) is standard. Here, we prove this results only to ensure the reader.

The following theorem is a local existence result (in the "Mild" sense) of the regularized system (P_ε)-(ID_ε). This result is achieved in a super-critical space. Here particularly we chose the space of functions $[C([0,T);X(\mathbb{R}))]^M$, where

$$X(\mathbb{R}) = \{u \in L^\infty(\mathbb{R}) \text{ such that } \partial_x u \in L^8(\mathbb{R})\}. \tag{2.12}$$

This space is a Banach space supplemented with the following norm

$$\|u\|_{X(\mathbb{R})} = \|u\|_{L^\infty(\mathbb{R})} + \|\partial_x u\|_{L^8(\mathbb{R})}.$$

Here the espace $L^p(\mathbb{R})$ with $p = 8$ will simplify later in Lemma 4.1 the Bootstrap argument to get smooth solution.

In this Section, we will prove the following

Theorem 2.2 (Local existence result)
For all initial data $u_0^\varepsilon \in [X(\mathbb{R})]^M$ there exists

$$T^\star = T^\star(M_0, \varepsilon) > 0,$$

such that the system (P_ε)-(ID_ε) admits solutions $u^\varepsilon \in [C([0,T^\star);X(\mathbb{R}))]^M$.

In order to do the proof of Theorem 2.2 in Subsection 2.2 we need to recall in the following Subsection some known results.

2.1 Useful results

***Lemma* 2.3 (Mild solution)**
Let $T > 0$, and $u^\varepsilon \in [C([0,T);X(\mathbb{R}))]^M$ be a solution of the following integral problem with $u^\varepsilon(t) = u^\varepsilon(t,\cdot)$:

$$u^\varepsilon(t) = S_\varepsilon(t)u_0^\varepsilon - \int_0^t S_\varepsilon(t-s)\left(a(u^\varepsilon(s)) \diamond \partial_x u^\varepsilon(s)\right) ds, \tag{IN_ε}$$

where $S_\varepsilon(t) = S_1(\varepsilon t)$ such that $S_1(t) = e^{t\Delta}$ is the heat semi-group. Then u^ε is a solution of the system (P_ε)-(ID_ε) in the sense of distributions.

For the proof of this lemma, we refer to Pazy [117, Th 5.2. Page 146].

Lemma* 2.4 *(Picard Fixed Point Theorem, see [85])
Let E be a Banach space, let $B : E \times E \longrightarrow E$ be a continuous map such that :

$$\|B(x,y)\|_E \leq \eta \|y\|_E \quad \text{for all} \quad x, y \in E,$$

where η is a positive given constant. Then, for every $x_0 \in E$, if

$$0 < \eta < 1,$$

the equation $x = x_0 + B(x,x)$ admits a solution in E.

In order to show the local existence of a solution for (IN_ε), we will apply Lemma 2.4 in the space $E = [L^\infty((0,T); X(\mathbb{R}))]^M$.

***Lemma* 2.5 (Time continuity)**
Let $T > 0$. If $u^\varepsilon \in [L^\infty((0,T); W^{1,p}(\mathbb{R}))]^M$, $1 \le p \le +\infty$, are solutions of integral problem (IN_ε), then $u^\varepsilon \in [C([0,T); W^{1,p}(\mathbb{R}))]^M$.

For the proof of Lemma 2.3, see A. Pazy [117, 7.3, Page 212].

***Lemma* 2.6 (Semi-group estimates)**
Let $1 \le p \le q \le +\infty$. Then for all $f \in L^p(\mathbb{R})$ and for all $t > 0$, we have the following estimates :

i) $\|S_\varepsilon(t)f\|_{L^q(\mathbb{R})} \le Ct^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R})}$,

ii) $\|\partial_x S_\varepsilon(t)f\|_{L^p(\mathbb{R})} \le Ct^{-\frac{1}{2}}\|f\|_{L^p(\mathbb{R})}$,

where $C = C(\varepsilon)$ is a positive constant depending on ε.

For the proof of this Lemma, see Pazy [117, Lemma 1.1.8, Th 6.4.5].

2.2 Proof of Theorem 2.2

Our goal is to show local existence of a solution of (P_ε) using the Picard fixed point Theorem. To be done according Lemma 2.3 it is enough to prove the local existence for the following equation :

$$\begin{aligned} u^\varepsilon(t) &= S_\varepsilon(t)u_0^\varepsilon - \int_0^t S_\varepsilon(t-s)\left(a(u^\varepsilon(s)) \diamond \partial_x u^\varepsilon(s)\right) ds, \\ &= S_\varepsilon(t)u_0^\varepsilon + B(u^\varepsilon, u^\varepsilon)(t), \end{aligned} \tag{2.13}$$

with $B(u,v)(t) = -\displaystyle\int_0^t S_\varepsilon(t-s)\left(a(u)(s) \diamond \partial_x v(s)\right) ds.$

If we estimate $B(u,v)$, we will obtain, for all $u, v \in [L^\infty((0,T); X(\mathbb{R}))]^M$, where $X(\mathbb{R})$ defined in (2.12), the following :

$$\begin{aligned} \|B(u,v)(t)\|_{[X(\mathbb{R})]^M} &= \left\| \int_0^t S_\varepsilon(t-s)\left(a(u(s)) \diamond \partial_x v(s)\right) ds, \right\|_{[L^\infty(\mathbb{R})]^M}, \\ &+ \left\| \int_0^t \partial_x S_\varepsilon(t-s)\left(a(u(s)) \diamond \partial_x v(s)\right) ds, \right\|_{[L^8(\mathbb{R})]^M}, \end{aligned} \tag{2.14}$$

where for a function $f = (f^1, \dots, f^M) \in [X(\mathbb{R})]^M$, we note here

$$\|f\|_{[X(\mathbb{R})]^M} = \sup_{i=1,\dots,M} \|f^i\|_{L^\infty(\mathbb{R})} + \sup_{i=1,\dots,M} \|\partial_x f^i\|_{L^8(\mathbb{R})}.$$

Using Lemma 2.6 (i) with $p = 8, q = \infty$ for the first term and Lemma 2.6 (ii) with $p = 8$ for the second term, we obtain that :

$$\|B(u,v)(t)\|_{[X(\mathbb{R})]^M} \leq C \int_0^t \frac{1}{(t-s)^{\frac{7}{16}}} \|a(u(s))\partial_x v(s)\|_{[L^2(\mathbb{R})]^M} \, ds,$$

$$+C \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|a(u(s))\partial_x v(s)\|_{[L^8(\mathbb{R})]^M} \, ds.$$

We use the Hölder inequality, and get, for all $0 < T \leq 1$:

$$\begin{aligned} \|B(u,v)(t)\|_{[X(\mathbb{R})]^M} &\leq CT^{\frac{1}{2}} \|\partial_x v\|_{[L^\infty((0,T);L^8(\mathbb{R}))]^M}, \\ &\leq CT^{\frac{1}{2}} \|v\|_{[L^\infty((0,T);X(\mathbb{R}))]^M}, \end{aligned} \tag{2.15}$$

where $C(M_0, \varepsilon)$. Moreover, we know by classical properties of heat semi-group (see A. Pazy [117]) :

$$\|S_\varepsilon(t)u_0^\varepsilon\|_{[L^\infty((0,T);X(\mathbb{R}))]^M} \leq \|u_0^\varepsilon\|_{[X(\mathbb{R})]^M}. \tag{2.16}$$

Now, taking

$$(T^\star)^{\frac{1}{2}} = \min\left(\frac{1}{2C}, 1\right), \tag{2.17}$$

we can easily verify that

$$C(T^\star)^{\frac{1}{2}} < 1.$$

By applying the Picard Fixed Point Theorem (Lemma 2.4) with $E = [L^\infty((0,T^\star); X(\mathbb{R}))]^M$, this proves the existence of a solution $u^\varepsilon \in [L^\infty((0,T^\star); X(\mathbb{R}))]^M$ for (2.13).

Then, according to Lemma 2.5, we deduce that the solution is indeed in $[C([0,T^\star); X(\mathbb{R}))]^M$.

This proves, by Lemma 2.3, the existence of a solution in $[C([0,T^\star); X(\mathbb{R}))]^M$, which satisfies the system (P_ε)-(ID_ε) in the sense of distributions. □

3 Global existence of the solutions of the approximated system

In this Section, we will prove the global existence of solution for the system (P_ε)-(ID_ε). Before going into the proof, we need the following lemma.

***Lemma* 3.1 (L^∞ bound)**
Let $T > 0$. If $u^\varepsilon \in [C([0,T); X(\mathbb{R}))]^M$ is a solution of system (P_ε)-(ID_ε) with initial data $u_0^\varepsilon \in X(\mathbb{R})$, then

$$\|u^\varepsilon\|_{[L^\infty([0,T)\times\mathbb{R})]^M} \leq \|u_0^\varepsilon\|_{[L^\infty(\mathbb{R})]^M}$$

The proof of this Lemma is a direct application of the Maximum Principle Theorem for parabolic equations (see Gilbarg-Trudinger [63, Th.3.1]).

Remark 3.2
Thanks to the previous Lemma, we notice that we can take the box U defined in (1.1) as the following

$$U = \Pi_{i=1}^M [-\|u_0^{\varepsilon,i}\|_{L^\infty(\mathbb{R})}, \|u_0^{\varepsilon,i}\|_{L^\infty(\mathbb{R})}].$$

For fixed ε, this definition guarantee that M_0 do not change in the course of time.

The result of this Section is the following.

Theorem 3.3 (Global existence)
Let $T > 0$ and $0 < \varepsilon \leq 1$. For initial data $u_0^\varepsilon \in [X(\mathbb{R})]^M$ satisfying $(H1)$ and $(H2)$. Then the system (P_ε)-(ID_ε), admits a solution $u^\varepsilon \in [C([0,T); X(\mathbb{R}))]^M$, with $u^\varepsilon(t,\cdot)$ satisfying $(H1)$ and $(H2)$ for all $t \in (0,T)$. Moreover, for all $t \in (0,T)$, we have the following inequalities :

$$\|u^{\varepsilon,i}(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \|u_0^{\varepsilon,i}\|_{L^\infty(\mathbb{R})}, \qquad \text{for } i = 1,\dots,M, \tag{3.18}$$

Proof of Theorem 3.3 :
We are going to prove that local in time solutions obtained by Theorem 2.2 can be extended to global solutions for the same system.

We argue by contradiction : assume that there exists a maximum time T_{max} such that, we have the existence of solutions of the system (P_ε)-(ID_ε) in the function space $[C([0,T_{max}); X(\mathbb{R}))]^M$.
For every small enough $\delta > 0$, we consider the system (P_ε) with the initial condition

$$u_0^{\varepsilon,\delta}(x) = u^\varepsilon(T_{max} - \delta, x).$$

From Theorem 2.2 to deduce that there exists a time $T^\star(M_0, \varepsilon)$, independent of δ (see Remark 3.2), such that the system (P_ε) with initial data $u_0^{\varepsilon,\delta}$ has a solution $u^{\varepsilon,\delta}$ on the time interval $[0, T^\star)$. Then for

$$T_0 = (T_{max} - \delta) + T^\star,$$

we extend u^ε on the time interval $[0, T_0)$ as follows,

$$\tilde{u}^\varepsilon(t,x) = \begin{cases} u^\varepsilon(t,x), & \text{for} \quad t \in [0, T_{max} - \delta], \\ u^{\varepsilon,\delta}(t,x), & \text{for} \quad t \in [T_{max} - \delta, T_0) \end{cases}$$

and we can check that $\tilde{u}^\varepsilon$ is a solution of (P_ε)-(ID_ε) on the time interval $[0, T_0)$. But from Lemma (3.1) we know that the time $T^\star$ is independent of δ (see Remark 3.2), which implies that $T_0 > T_{max}$ and so a contradiction.
The inequalities (3.18) is a consequence of Lemma 3.1. □

4 Properties of the solutions of the approximated system

In this section, we are going to prove that the solution of (P_ε)-(ID_ε) obtained by Theorem 2.2 is smooth and monotone.

***Lemma* 4.1 (Smoothness of the solution)**
Let $T > 0$. For all initial data $u_0^\varepsilon \in [X(\mathbb{R})]^M$, where $\partial_x u_0^\varepsilon \in [W^{m,p}(\mathbb{R})]^M$ for all $m \in \mathbb{N}$, $1 \le p \le +\infty$.

If u^ε is a solution of the system (P_ε)-(ID_ε), such that $u^\varepsilon \in [C([0,T); X(\mathbb{R}))]^M$ and $\partial_x u^\varepsilon \in [L^\infty((0,T); L^1(\mathbb{R}))]^M$, then $u^\varepsilon \in [C^\infty([0,T) \times \mathbb{R})]^M$ and satisfies,

$$u^\varepsilon \in [W^{m,p}((0,T) \times \mathbb{R})]^M, \quad \textit{for all } 1 < p \le +\infty \textit{ and } m \in \mathbb{N} \setminus \{0\}, \tag{4.19}$$

Proof of Lemma 4.1

Step 1 (Initialization of the Bootstrap) :

For the sake of simplicity, we will set

$$F[u^\varepsilon] = -a(u^\varepsilon) \diamond \partial_x u^\varepsilon.$$

From the fact that $u^\varepsilon \in [C([0,T); X(\mathbb{R}))]^M$ and $\partial_x u^\varepsilon \in [L^\infty((0,T); L^1(\mathbb{R}))]^M$, we deduce that $\partial_x u^\varepsilon, \quad F[u^\varepsilon] \in [L^1((0,T) \times \mathbb{R})]^M \cap [L^8((0,T) \times \mathbb{R})]^M$, which proves by interpolation that

$$\partial_x u^\varepsilon, \quad F[u^\varepsilon] \in [L^p((0,T)\times\mathbb{R})]^M \quad \text{for all } 1 \le p \le 8. \tag{4.20}$$

Because u^ε is a solution of (P_ε), we see that

$$\partial_t u^\varepsilon - \varepsilon \partial_{xx} u^\varepsilon = F[u^\varepsilon], \tag{4.21}$$

$$\partial_{tx} u^\varepsilon - \varepsilon \partial_{xxx} u^\varepsilon = \partial_x F[u^\varepsilon]. \tag{4.22}$$

Applaying the classical regularity theory of heat equations on (4.21), we deduce that :

$$\partial_t u^\varepsilon \quad \text{and} \quad \partial_{xx} u^\varepsilon \in [L^p((0,T)\times\mathbb{R})]^M, \quad \text{for all } 1 < p \le 8. \tag{4.23}$$

For more details, see Ladyzenskaja [96, Theorem 9.1]. But we know that

$$\partial_x F[u^\varepsilon] = -a(u^\varepsilon) \diamond \partial_{xx} u^\varepsilon - Da(u^\varepsilon)\partial_x u^\varepsilon \diamond \partial_x u^\varepsilon \tag{4.24}$$

We notice that thanks to this better regularity on u^ε ((4.20) and (4.23), and by the Hölder inequality we can easily prove that

$$\partial_x F[u^\varepsilon] \in [L^p((0,T)\times\mathbb{R})]^M \quad \text{for all } 1 < p \le 4.$$

Now, we apply again the classical regularity theory of heat equations on (4.22), to deduce that :

$$\partial_{tx} u^\varepsilon \quad \text{and} \quad \partial_{xxx} u^\varepsilon \in [L^p((0,T)\times\mathbb{R})]^M, \quad \text{for all } 1 < p \le 4. \tag{4.25}$$

We know that

$$\partial_t F[u^\varepsilon] = -a(u^\varepsilon) \diamond \partial_{tx} u^\varepsilon - Da(u^\varepsilon)\partial_t u^\varepsilon \diamond \partial_x u^\varepsilon \tag{4.26}$$

Thanks this previous regularity on u^ε, we obtain by the Hölder inequality that

$$\partial_t F[u^\varepsilon] \in [L^p((0,T)\times\mathbb{R})]^M \quad \text{for all } 1 < p \le 4.$$

Which gives that

$$\partial_x u^\varepsilon, \quad F[u^\varepsilon] \in \left[W^{1,p}((0,T)\times\mathbb{R})\right]^M \quad \text{for all } 1 < p \le 4,$$

and by the Sobolev embedding that $\partial_x u^\varepsilon \in [L^p((0,T)\times\mathbb{R})]^M$ for all $1 < p \le \infty$.

Step 2 (Recurrence) :

Now, we use the same steps, we can prove by recurrence that for all $m \in \mathbb{N}$ if,

$$(H_m) \quad \left| \begin{array}{l} \partial_x u^\varepsilon \in [L^\infty((0,T)\times\mathbb{R})]^M, \\ \\ \partial_x u^\varepsilon, \;\; F[u^\varepsilon] \in [W^{m,p}((0,T)\times\mathbb{R})]^M \quad \text{for all } 1<p\leq 4, \end{array} \right.$$

then

$$(H_m) \Rightarrow (H_{m+1}).$$

Indeed, as in (4.23) we can deduce here that

$$\partial_t u^\varepsilon \quad \text{and} \quad \partial_{xx} u^\varepsilon \in [W^{m,p}((0,T)\times\mathbb{R})]^M, \quad \text{for all } 1<p\leq 4, \tag{4.27}$$

and From (4.24), because $\partial_x u^\varepsilon \in [L^\infty((0,T)\times\mathbb{R})]^M$, we can obtain here that

$$\partial_x F[u^\varepsilon] \in [W^{m,p}((0,T)\times\mathbb{R})]^M \quad \text{for all } 1<p\leq 4.$$

Which proves that, as in (4.25) that

$$\partial_{tx} u^\varepsilon \quad \text{and} \quad \partial_{xxx} u^\varepsilon \in [W^{m,p}((0,T)\times\mathbb{R})]^M, \quad \text{for all } 1<p\leq 4, \tag{4.28}$$

and From (4.26), we deduce that

$$\partial_t F[u^\varepsilon] \in [W^{m,p}((0,T)\times\mathbb{R})]^M \quad \text{for all } 1<p\leq 4,$$

and then

$$\partial_x u^\varepsilon, \;\; F[u^\varepsilon] \in \left[W^{m+1,p}((0,T)\times\mathbb{R})\right]^M \quad \text{for all } 1<p\leq 4,$$

Which proves by the Sobolev embedding the results. □

***Lemma* 4.2 (Classical Maximum Principle)**
Let $T>0$. For all initial data $u_0^\varepsilon \in [X(\mathbb{R})]^M$, where $\partial_x u_0^\varepsilon \in [W^{m,p}(\mathbb{R})]^M$ for all $m\in\mathbb{N}$, $1\leq p\leq +\infty$, and satisfying $(H3)$.

If u^ε is a solution of the system (P_ε)-(ID_ε), such that $u^\varepsilon \in [C([0,T);X(\mathbb{R}))]^M$ and $\partial_x u^\varepsilon \in [L^\infty((0,T);L^1(\mathbb{R}))]^M$, then we have for $i=1,\dots,M$, $\partial_x u^{\varepsilon,i}\geq 0$ on $(0,T)\times\mathbb{R}$.

Proof of Lemma 4.2
We first derive with respect to x the system (P_ε)-(ID_ε), and get for $w^\varepsilon = (w^{\varepsilon,i})_{i=1,\dots,M} = \partial_x u^\varepsilon$

$$\partial_t w^\varepsilon - \varepsilon\partial_{xx} w^\varepsilon + a(u^\varepsilon)\diamond\partial_x w^\varepsilon + Da(u)w^\varepsilon\diamond w^\varepsilon = 0.$$

Since $u^\varepsilon \in [C^\infty([0,T)\times\mathbb{R})]^M$, we see, for $i=1,\dots,M$, that $w^{\varepsilon,i}$ is smooth and satisfies $w^{\varepsilon,i}(0,x) = \partial_x u_0^{\varepsilon,i} \geq 0$. From the classical maximum principle we deduce that $w^{\varepsilon,i}\geq 0$ on $[0,T)\times\mathbb{R}$. □

Remark 4.3 **(L^1 uniform estimate on $\partial_x u^\varepsilon$)**
Because $\partial_x u^{\varepsilon,i} \geq 0$, for $i = 1, \ldots, M$, we deduce from Lemma 3.1 that :

$$\|\partial_x u^\varepsilon\|_{[L^\infty([0,T);L^1(\mathbb{R}))]^M} \leq 2\,\|u^\varepsilon\|_{[L^\infty([0,T)\times\mathbb{R})]^M} \leq 2\|u_0^\varepsilon\|_{[L^\infty(\mathbb{R})]^M}. \tag{4.29}$$

Corollary 4.4 **(global existence of nondecreasing smooth solutions)**
Let $T > 0$. The solution given in Theorem 2.2 can be chosen such that $u^\varepsilon = (u^{\varepsilon,i})_{i=1,\ldots,M}$ smooth, satisfies (4.19) and for each $i = 1, \ldots, M$, $\partial_x u^{\varepsilon,i} \geq 0$ on $(0,T) \times \mathbb{R}$.

The proof of Corollary 4.4 is a consequence of Theorem 2.2 and Lemmata 4.1, 4.2 and Remark 4.3.

5 ε-Uniform *a priori* estimates

In this Section, we show some ε-uniform estimates on the solutions of the system (P_ε)-(ID_ε). These estimates will be used in Section 6 for the passage to the limit as ε tends to zero.

***Lemma* 5.1** **(L^∞ bound on u^ε and L^1 bound on $\partial_x u^\varepsilon$)**
Let $T > 0$, $0 < \varepsilon \leq 1$ and function $u_0 \in [L^\infty(\mathbb{R})]^M$ satisfying $(H3)$. Then the solution of the system (P_ε)-(ID_ε) given in Theorem 3.3 with initial data $u_0^\varepsilon = u_0 * \eta_\varepsilon$, satisfies the following ε-uniform estimates :

$(E1)$ $\|u^\varepsilon\|_{[L^\infty((0,T)\times\mathbb{R})]^M} \leq \|u_0\|_{[L^\infty(\mathbb{R})]^M}$,

$(E2)$ $\|\partial_x u^\varepsilon\|_{[L^\infty((0,T),L^1(\mathbb{R}))]^M} \leq 2\,\|u_0\|_{[L^\infty(\mathbb{R})]^M}$,

Proof of Lemma 5.1 :
First, we remark that if $\partial_x u_0 \geq 0$, then $\partial_x u_0^\varepsilon = (\partial_x u_0) * \eta_\varepsilon(x) \geq 0$ (because η is positive). The fact that $u_0 \in [L^\infty(\mathbb{R})]^M$ and $\partial_x u_0 \geq 0$, we obtain that $\partial_x u_0 \in [L^1(\mathbb{R})]^M$.
By classical properties of the mollifier $(\eta_\varepsilon)_\varepsilon$ we know that if $u_0 \in [L^\infty(\mathbb{R})]^M$ and $\partial_x u_0 \in [L^1(\mathbb{R})]^M$ we have $u_0^\varepsilon \in [X(\mathbb{R})]^M$ and $\partial_x u_0^\varepsilon \in [W^{m,p}(\mathbb{R})]^M$ for all $m \in \mathbb{N}$, $1 \leq p \leq +\infty$.

Now, we use Lemma 3.1 and Remark 4.3, we deduce by the classical properties of the mollifier $(E1)$ and $(E2)$.

Before going into the proof of the gradient entropy inequality defined in (5.30), we announce the main idea of this new gradient entropy estimate. Now, let us set for $w \geq 0$ the entropy function

$$\bar{f}(w) = w \ln w.$$

For any *non-negative* test function $\varphi \in C_c^1(\mathbb{R} \times [0, +\infty))$, let us define the following *"gradient entropy"* with $w^i := \partial_x u^i$:

$$\bar{N}(t) = \int_{\mathbb{R}} \varphi \left(\sum_{i=1,\dots,M} \bar{f}(w^i) \right) \, dx.$$

It is very natural to introduce such quantity $\bar{N}(t)$ which in the case $\varphi \equiv 1$, appears to be nothing else than the total entropy of the system of M type of particles of non-negative densities w^i. Then it is formally possible to deduce from (P) the equality in the following new *gradient entropy inequality* for all $t \geq 0$

$$\frac{d\bar{N}}{dt}(t) + \int_{\mathbb{R}} \varphi \left(\sum_{i,j=1,\dots,M} a^i_{,j} w^i w^j \right) \, dx \leq R(t) \qquad \text{for} \quad t \geq 0, \tag{5.30}$$

with the rest

$$R(t) = \int_{\mathbb{R}} \left\{ (\partial_t \varphi) \left(\sum_{i=1,\dots,M} \bar{f}(w^i) \right) + (\partial_x \varphi) \left(\sum_{i=1,\dots,M} a^i \bar{f}(w^i) \right) \right\} \, dx,$$

where we only show the dependence on t in the integrals. We remark in particular that this rest is formally equal to zero if $\varphi \equiv 1$.

To guarantee the existence of continuous solutions, we assumed in $(H2)$ a sign on the left hand side of inequality (5.30).
For we return this previous calculate more rigorous, we prove actually the following gradient entropy inequality

Proposition 5.2 (Gradient entropy inequality)
Let $T > 0$, $0 < \varepsilon \leq 1$ and function $u_0 \in [L^\infty(\mathbb{R})]^M$ satisfying $(H3)$. We consider the solution u^ε of the system (P_ε)-(ID_ε) given in Theorem 3.3 with initial data $u_0^\varepsilon = u_0 * \eta_\varepsilon$,. Then, there exists a constant $C(T, M, M_1, \|u_0\|_{[L^\infty(\mathbb{R})]^M}, \|\partial_x u_0\|_{[L \log L(\mathbb{R})]^M}$ such that

$$N(t) + \int_0^t \int_{\mathbb{R}} \sum_{i,j=1,\dots,M} a^i_{,j}(u^\varepsilon) w^{\varepsilon,i} w^{\varepsilon,j} \leq C, \quad \textit{with} \quad N(t) = \int_{\mathbb{R}} \sum_{i=1,\dots,M} f(w^{\varepsilon,i}) dx. \tag{5.31}$$

where $w^\varepsilon = (w^{\varepsilon,i})_{i=1,\dots,M} = \partial_x u^\varepsilon$ and f is defined in (1.4).

For the proof of Proposition 5.2 we need the following Lemma :

***Lemma* 5.3 ($L\log L$ Estimate)**
Let $(\eta_\varepsilon)_\varepsilon$ be a non-negative mollifier, f is the function defined in (1.4) and $h \in L^1(\mathbb{R})$ is a non-negative function. Then
i) $\displaystyle\int_{\mathbb{R}} f(h) < +\infty$ *if and only if* $h \in L\log L(\mathbb{R})$.
ii) If $h \in L\log L(\mathbb{R})$ the function $h_\varepsilon = h * \eta_\varepsilon \in L\log L(\mathbb{R})$ satisfies

$$\|h - h_\varepsilon\|_{L\log L(\mathbb{R})} \to 0 \qquad as \qquad \varepsilon \to 0.$$

The proof of (i) is trivial, for the proof of (ii) see R. A. Adams [2, Th 8.20] for the proof of this Lemma.
Proof of Proposition 5.2 :
Remark first that the quantity $N(t)$ is well-defined because $w^\varepsilon \in [L^\infty((0,T);L^1(\mathbb{R}))]^M \cap [L^\infty((0,T);L^8(\mathbb{R}))]^M$ (by Theorem 2.2 and Corollary 4.4) and we have the general inequality $\frac{-1}{e} \leq w\log w \leq w^2$ for all $w \geq 0$.

From Theorem 4.4 we know that $w^{\varepsilon,i}$ and smooth non-negative function. Now, we derive $N(t)$ with respect to t, this is well-defined because for $i = 1,\dots,M$, we have $\left|\displaystyle\int_{w^{\varepsilon,i}\geq\frac{1}{e}}\right| \leq e\|w^{\varepsilon,i}\|_{L^\infty((0,T);L^1(\mathbb{R}))}$ and for all $m \in \mathbb{N}$, $w^{\varepsilon,i} \in W^{m,\infty}((0,T)\times\mathbb{R})$ (see (4.19)).

Finally, we get that,

$$\begin{aligned}
\frac{d}{dt}N(t) &= \int_{\mathbb{R}} \sum_{i=1,\dots,M} f'(w^{\varepsilon,i})(\partial_t w^{\varepsilon,i}),\\
&= \int_{\mathbb{R}} \sum_{i=1,\dots,M} f'(w^{\varepsilon,i})\partial_x\left(-a^i(u^\varepsilon)w^{\varepsilon,i} + \varepsilon\partial_x w^{\varepsilon,i}\right),\\
&= \overbrace{\int_{\mathbb{R}} \sum_{i=1,\dots,M} a^i(u^\varepsilon)w^{\varepsilon,i}f''(w^{\varepsilon,i})\partial_x w^{\varepsilon,i}}^{J_1} \overbrace{- \varepsilon\int_{\mathbb{R}} \sum_{i=1,\dots,M} \left(\partial_x w^{\varepsilon,i}\right)^2 f''(w^{\varepsilon,i})}^{J_2}
\end{aligned}$$

But, it is easy to check that

$$f'(x) = \begin{cases} \ln(x)+1 & \text{if} \quad x \geq 1/e,\\ 0 & \text{if} \quad 0 \leq x \leq 1/e, \end{cases} \qquad \text{and} \quad f''(x) = \begin{cases} \frac{1}{x} & \text{if} \quad x \geq 1/e,\\ 0 & \text{if} \quad 0 \leq x \leq 1/e. \end{cases}$$

This proves that $J_2 \leq 0$. To control J_1, we rewrite it under the following form

$$J_1 = \int_{\mathbb{R}} \sum_{i=1,\dots,M} a^i(u^\varepsilon) g'(w^{\varepsilon,i}) \partial_x w^{\varepsilon,i},$$

where

$$g(x) = \begin{cases} x - \frac{1}{e} & \text{if} \quad x \geq 1/e, \\ 0 & \text{if} \quad 0 \leq x \leq 1/e, \end{cases}$$

Then, we deduce that

$$\begin{aligned} J_1 &= \int_{\mathbb{R}} \sum_{i=1,\dots,M} a^i(u^\varepsilon) \partial_x (g(w^{\varepsilon,i})) \\ &= -\int_{\mathbb{R}} \sum_{i,j=1,\dots,M} a^i_{,j}(u^\varepsilon) w^{\varepsilon,j} g(w^{\varepsilon,i}), \\ &= \overbrace{-\int_{\mathbb{R}} \sum_{i,j=1,\dots,M} a^i_{,j}(u^\varepsilon) w^{\varepsilon,j} w^{\varepsilon,i}}^{J_{11}} \overbrace{-\int_{\mathbb{R}} \sum_{i,j=1,\dots,M} a^i_{,j}(u^\varepsilon) w^{\varepsilon,j} (g(w^{\varepsilon,i}) - w^{\varepsilon,i})}^{J_{12}}, \end{aligned}$$

From $(H2)$, we know that $J_{11} \leq 0$. We use the fact that $|g(x) - x| \leq \frac{1}{e}$ for all $x \geq 0$ and $(H1)$, to deduce that

$$\begin{aligned} |J_{12}| &\leq \tfrac{1}{e} M^2 M_1 \, \|w^{\varepsilon,i}\|_{[L^\infty((0,T),L^1(\mathbb{R}))]^M} \\ &\leq \tfrac{2}{e} M^2 M_1 \|u_0\|_{[L^\infty(\mathbb{R})]^M} \end{aligned}$$

where we have use Lemma 5.1 $(E2)$ in the last line. Finally, we deduce that, there exists a positive constant $C(\|u_0\|_{[L^\infty(\mathbb{R})]^M}, M_1, M)$ independent of ε such that

$$\begin{aligned} \frac{d}{dt} N(t) &\leq J_{11} + J_{12} + J_2 \\ &\leq J_{11} + C. \end{aligned}$$

Integrating in time we get by Lemma 5.3, there exists a another positive constant $C(T, M, M_1, \|u_0\|_{[L^\infty(\mathbb{R})]^M}, \|\partial_x u_0\|_{[L \log L(\mathbb{R})]^M})$ independent of ε such that

$$N(t) + \int_0^t \int_{\mathbb{R}} \sum_{i,j=1,\dots,M} a^i_{,j}(u^\varepsilon) w^{\varepsilon,j} w^{\varepsilon,i} \leq CT + N(0) \leq C.$$

□

***Lemma* 5.4 ($W^{-1,1}$ estimate on the time derivatives of the solutions)**
Let $T > 0$, $0 < \varepsilon \leq 1$ and function $u_0 \in [L^\infty(\mathbb{R})]^M$ satisfying $(H3)$. Then the solution of the system (P_ε)-(ID_ε) given in Theorem 3.3 with initial data $u_0^\varepsilon = u_0 * \eta_\varepsilon$, satisfies the following ε-uniform estimates :

$$\|\partial_t u^\varepsilon\|_{[L^2((0,T);W^{-1,1}(\mathbb{R}))]^M} \leq C\left(1 + \|u_0\|^2_{[L^\infty(\mathbb{R})]^M}\right).$$

where $W^{-1,1}(\mathbb{R})$ is the dual of the space $W^{1,\infty}(\mathbb{R})$.

Proof of Lemma 5.4 :
The idea to bound $\partial_t u^\varepsilon$ is simply to use the available bounds on the right hand side of the equation (P_ε).
We will give a proof by duality. We multiply the equation (P_ε) by $\phi \in [L^2((0,T), W^{1,\infty}(\mathbb{R}))]^M$ and integrate on $(0,T) \times \mathbb{R}$, which gives

$$\int_{(0,T)\times\mathbb{R}} \phi\, \partial_t u^\varepsilon = \overbrace{\varepsilon \int_{(0,T)\times\mathbb{R}} \phi\, \partial^2_{xx} u^\varepsilon}^{I_1} \overbrace{- \int_{(0,T)\times\mathbb{R}} \phi\, a(u^\varepsilon) \diamond \partial_x u^\varepsilon}^{I_2}.$$

We integrate by parts the term I_1, and obtain that for $0 < \varepsilon \leq 1$:

$$\begin{aligned} |I_1| \leq \left|\int_{(0,T)\times\mathbb{R}} \partial_x\phi \partial_x u^\varepsilon\right| &\leq T\|\partial_x\phi\|_{[L^2((0,T),L^\infty(\mathbb{R}))]^M} \|\partial_x u^\varepsilon\|_{[L^2((0,T),L^1(\mathbb{R}))]^M}, \\ &\leq 2T^{\frac{3}{2}} \|\phi\|_{[L^2((0,T),W^{1,\infty}(\mathbb{R}))]^M} \|u_0\|_{[L^\infty(\mathbb{R})]^M}, \end{aligned} \tag{5.32}$$

here, we have used the inequality

$$\|\partial_x u^\varepsilon\|_{[L^2([0,T);L^1(\mathbb{R}))]^M} \leq 2T^{\frac{1}{2}} \|u_0\|_{[L^\infty(\mathbb{R})]^M}, \tag{5.33}$$

which follows from estimate (4.29) for bounded and nondecreasing function u^ε. Similarly, for the term I_2, we have :

$$\begin{aligned} |I_2| &\leq M_0 \|u\|_{[L^\infty((0,T)\times\mathbb{R})]^M} \|\phi\|_{[L^2((0,T),L^\infty(\mathbb{R}))]^M} \|\partial_x u^\varepsilon\|_{[L^2((0,T),L^1(\mathbb{R}))]^M}, \\ &\leq 2T^{\frac{1}{2}} M_0 \|u_0\|^2_{[L^\infty(\mathbb{R})]^M} \|\phi\|_{[L^2((0,T),W^{1,\infty}(\mathbb{R}))]^M}. \end{aligned} \tag{5.34}$$

Finally, collecting (5.32) and (5.34), we get that there exists a constant $C = C(T, M_0)$ independent of $0 < \varepsilon \leq 1$ such that :

$$\left|\int_{(0,T)\times\mathbb{R}} \phi \partial_t u^\varepsilon\right| \leq C\left(1 + \|u_0\|^2_{[L^\infty(\mathbb{R})]^M}\right) \|\phi\|_{[L^2((0,T),W^{1,\infty}(\mathbb{R}))]^M}$$

which gives the announced result where we use that $L^2((0,T), W^{-1,1}(\mathbb{R}))$ is the dual of $L^2((0,T), W^{1,\infty}(\mathbb{R}))$ (see Cazenave and Haraux [28, Th 1.4.19, Page 17]). □

Corollary 5.5 (ε-Uniform estimates)
Let $T > 0$, $0 < \varepsilon \leq 1$ and function $u_0 \in [L^\infty(\mathbb{R})]^M$ satisfying $(H1)$ and $(H2)$. Then the solution of the system (P_ε)-(ID_ε) given in Theorem 3.3 with initial data $u_0^\varepsilon = u_0 * \eta_\varepsilon$, satisfies the following ε-uniform estimates :

$$\|\partial_x u^\varepsilon\|_{[L^\infty((0,T);L\log L(\mathbb{R}))]^M} + \|u^\varepsilon\|_{[L^\infty((0,T)\times\mathbb{R})]^M} + \|\partial_t u^\varepsilon\|_{[L^2((0,T);W^{-1,1}(\mathbb{R}))]^M} \leq C.$$

where $C = C(T, M, M_0, M_1 \|u_0\|_{[L^\infty(\mathbb{R})]^M}, \|\partial_x u_0\|_{[L\log L(\mathbb{R})]^M})$.

We can easily prove this Corollary collecting Lemmata 5.1, 5.4 and 5.3 and Proposition 5.2.

6 Passage to the limit and the proof of Theorem 1.1

In this section, we prove that the system (P)-(ID) admits solutions u in the distributional sense. They are the limits of u^ε given by Theorem 3.3 when $\varepsilon \to 0$. To do this, we will justify the passage to the limit as ε tends to 0 in the system (P_ε)-(ID_ε) by using some compactness tools that are presented in a first Subsection.

6.1 Preliminary results

First, for all I open interval of $\mathbb{R}$, we denote by

$$L\log L(I) == \left\{ f \in L^1(I) \text{ such that } \int_I |f| \ln(1+|f|) < +\infty \right\}.$$

***Lemma* 6.1 (Compact embedding)**
Let I an open and bounded interval of $\mathbb{R}$. If we denote by

$$W^{1,L\log L}(I) = \{u \in L^1(I) \text{ such that } \partial_x u \in L\log L(I)\}.$$

Then the following injection :

$$W^{1,L\log L}(I) \hookrightarrow C(I),$$

is compact.

For the proof of this Lemma see R. A. Adams [2, Th 8.32].

***Lemma* 6.2 (Simon's Lemma)**
Let X, B, Y be three Banach spaces, such that

$X \hookrightarrow B$ with compact embedding and $B \hookrightarrow Y$ with continuous embedding.

Let $T > 0$. If $(u^\varepsilon)_\varepsilon$ is a sequence such that,

$$\|u^\varepsilon\|_{L^\infty((0,T);X)} + \|u^\varepsilon\|_{L^\infty((0,T);B)} + \|\partial_t u^\varepsilon\|_{L^q((0,T);Y)} \leq C,$$

where $q > 1$ and C is a constant independent of ε, then $(u^\varepsilon)_\varepsilon$ is relatively compact in $C((0,T);B)$.

For the proof, see J. Simon [125, Corollary 4, Page 85].

In order to show the existence of solution system (P) in Subsection 6.2, we will apply this lemma to each scalar component in the particular case where $X = W^{1,\log}(I)$, $B = L^\infty(I)$ and $Y = W^{-1,1}(I) := (W^{1,\infty}(I))'$.

We denote by $K_{exp}(I)$ the class of all measurable function u, defined on I, for which,

$$\int_I \left(e^{|u|} - 1\right) < +\infty.$$

The space $EXP(I)$ is defined to be the linear hull of $K_{exp}(I)$. This space is supplemented with the following Luxemburg norm (see Adams [2, (13), Page 234]) :

$$\|u\|_{EXP(I)} = \inf\left\{\lambda > 0 : \int_I \left(e^{\frac{|u|}{\lambda}} - 1\right) \leq 1\right\},$$

Let us recall some useful properties of this space.

***Lemma* 6.3 (Weak star topology in $L\log L$)**
Let $E_{exp}(I)$ be the closure in $EXP(I)$ of the space of functions bounded on I. Then $E_{exp}(I)$ is a separable Banach space which verifies,

i) $$L\log L(I) \text{ is the dual space of } E_{exp}(I).$$

ii) $$L^\infty(I) \hookrightarrow E_{exp}(I).$$

For the proof, see Adams [2, Th 8.16, 8.18, 8.20].

***Lemma* 6.4 (Generalized Hölder inequality, Adams [2, 8.11, Page 234])**
Let $f \in EXP(I)$ and $g \in L\log L(I)$. Then $fg \in L^1(I)$, with

$$\|fg\|_{L^1(I)} \leq 2\|f\|_{EXP(I)}\|g\|_{L\log L(I)}.$$

The following Lemma, we allow to define later the restriction of a function $f \in W^{-1,1}(\mathbb{R})$ on all open interval I of $\mathbb{R}$.

***Lemma* 6.5 (Extension)**
For all open interval I of $\mathbb{R}$, there exists a linear and continuous operator of extension $P : W^{1,\infty}(I) \to W^{1,\infty}(\mathbb{R})$ such that

i) $Pu_{|_I} = u$ for $u \in W^{1,\infty}(I)$.

ii) $\|Pu\|_{W^{1,\infty}(\mathbb{R})} \leq \|u\|_{W^{1,\infty}(I)}$ for $u \in W^{1,\infty}(I)$.

for the proof of this Lemma see for instance Brezis [21, Th.8.5].

Thanks this Lemma, we can notice that, if $f \in W^{-1,1}(\mathbb{R})$, where $W^{-1,1}(\mathbb{R}) := (W^{1,\infty}(\mathbb{R}))'$, we can define, for all open interval I of $\mathbb{R}$, the function $f_{|_I}$ as the following

$$< f_{|_I}, h >_{W^{-1,1}(I),W^{1,\infty}(I)} = < f, Ph >_{W^{-1,1}(\mathbb{R}),W^{1,\infty}(\mathbb{R})} .$$

for all $h \in W^{1,\infty}(I)$.

6.2 Proof of Theorem 1.1

Step 1 (Existence) :

First, by Corollary 5.5 we know that for any $T > 0$, the solutions u^ε of the system (P_ε)-(ID_ε) obtained with the help of Theorem 3.3, are ε-uniformly bounded in $[L^\infty((0,T) \times \mathbb{R})]^M$. Hence, as ε goes to zero, we can extract a subsequence still denoted by u^ε, that converges weakly-$\star$ in $[L^\infty((0,T) \times \mathbb{R})]^M$ to some limit u. Then we want to show that u is a solution of the system (P)-(ID). Indeed, since the passage to the limit in the linear terms is trivial in $[\mathcal{D}'((0,T) \times \mathbb{R})]^M$, it suffices to pass to the limit in the non-linear term,

$$a(u^\varepsilon) \diamond \partial_x u^\varepsilon.$$

According to Corollary 5.5 we know that for all open and bounded interval I of $\mathbb{R}$ there exists a constant C independent on ε such that :

$$\|u^\varepsilon\|_{\left[L^\infty((0,T);W^{1,L\log L}(I))\right]^M} + \|u^\varepsilon\|_{[L^\infty((0,T)\times I)]^M} + \|\partial_t u^\varepsilon\|_{[L^2((0,T);W^{-1,1}(I))]^M} \leq C.$$

From the compactness of $W^{1,L\log L}(I) \hookrightarrow L^\infty(I)$ (see Lemma 6.3 (i)), we can apply Simon's Lemma (i.e. Lemma 6.2), with $X = \left[W^{1,L\log L}(I)\right]^M$, $B = [L^\infty(I)]^M$ and $Y = [W^{-1,1}(I)]^M$, which shows that

$$u^\varepsilon \text{ is relatively compact in in } [L^\infty((0,T)\times I)]^M \hookrightarrow [L^1((0,T);L^\infty(I))]^M . \quad (6.35)$$

Then form continuous injection of $L^\infty(I) \hookrightarrow E_{exp}(I)$ (see Lemma 6.3 (ii)), we deduce that,

$$u^\varepsilon \text{ is relatively compact in } [L^1((0,T);E_{exp}(\Omega))]^M. \quad (6.36)$$

On the other hand, by Corollary 5.5, we notice that $\partial_x u^\varepsilon$ is ε-uniformly bounded in $[L^\infty((0,T);L\log L(I))]^M$. Moreover, the space $[L^\infty((0,T);L\log L(I))]^M$ is the dual space of $[L^1((0,T);E_{exp}(I))]^M$, because $L\log L(I)$ is the dual space of $E_{exp}(I)$ (see Lemma 6.3 (ii) and Cazenave, Haraux [28, Th 1.4.19, Page 17]). Then, up to a subsequence

$$\partial_x u^\varepsilon \to \partial_x u \text{ weakly-}\star \text{ in } [L^\infty((0,T);L\log L(I))]^M . \quad (6.37)$$

Form (6.36) and (6.37), we see that we can pass to the limit in the non-linear term in the sense

$$\left[L^1((0,T);E_{exp}(I))\right]^M - strong \ \times \ [L^\infty((0,T);L\log L(I))]^M - weak-\star.$$

Because this is true for any bounded open interval I and for any $T>0$, we deduce that,

$$a(u^\varepsilon)\diamond\partial_x u^\varepsilon \to a(u)\diamond\partial_x u \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R})$$

Consequently, we can pass to the limit in (P_ε) and get that,

$$\partial_t u + a(u)\diamond\partial_x u = 0 \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}).$$

This solution u is also satisfy the following estimates (see for instance Brezis [21, Prop. 3.12]) :

$(E1')$ $\|\partial_x u\|_{[L^\infty((0,T);L\log L(\mathbb{R}))]^M} \le \liminf \|\partial_x u^\varepsilon\|_{[L^\infty((0,T);L\log L(\mathbb{R}))]^M} \le C,$

$(E2')$ $\|u\|_{[L^\infty((0,T)\times\mathbb{R})]^M} \le \liminf \|u^\varepsilon\|_{[L^\infty((0,T)\times\mathbb{R})]^M} \le \|u_0\|_{[L^\infty(\mathbb{R})]^M}$,

At this stage we remark that, thanks to these two estimates we obtain that $(a(u)\diamond\partial_x u) \in [L^\infty((0,T);L\log L(\mathbb{R}))]^M$, which gives, since $\partial_t u = -a(u)\diamond\partial_x u$, that $\partial_t u \in [L^\infty((0,T);L\log L(\mathbb{R}))]^M$, and then $u \in [C([0,T);L\log L(\mathbb{R}))]^M$.

Step 2 (The initial conditions) :

It remains to prove that the initial conditions (ID) coincides with $u(\cdot,0)$. Indeed, by Corollary 5.5, we see that, for all open bounded interval I of $\mathbb{R}$, u^ε is ε-uniformly bounded in

$$\left[W^{1,2}((0,T);W^{-1,1}(I))\right]^M \hookrightarrow \left[C^{\frac{1}{2}}([0,T);W^{-1,1}(I))\right]^M,$$

where $W^{-1,1}(I)$ is the dual of $W^{1,\infty}(I)$. It follows that, there exists a constant C independent on ε, such that, for all $t,s\in[0,T)$:

$$\|u^\varepsilon(t)-u^\varepsilon(s)\|_{[W^{-1,1}(I)]^M}\leq C|t-s|^{\frac{1}{2}}.$$

In particular if we set $s=0$, we have :

$$\|u^\varepsilon(t)-u_0^\varepsilon\|_{[W^{-1,1}(I)]^M}\leq Ct^{\frac{1}{2}}. \tag{6.38}$$

Now we pass to the limit in (6.38). Indeed, the functions u^ε and u_0^ε are ε-uniformly bounded in $[W^{1,2}((0,T);W^{-1,1}(I))]^M$ and $[W^{-1,1}(I)]^M$ respectively. Moreover we know that $u^\varepsilon-u_0^\varepsilon$ converges weakly-$\star$ in $[L^\infty((0,T)\times I)]^M$ to $u-u_0$.

Therefore, we can extract a subsequence still denoted by $u^\varepsilon-u_0^\varepsilon$, that weakly-$\star$ converges in $[W^{1,2}((0,T);W^{-1,1}(I))]^M$ to $u-u_0$. In particular this subsequence converges, for all $t\in(0,T)$, weakly-$\star$ in $[L^\infty((0,t);W^{-1,1}(I))]^M$, and consequently it verifies (see for instance Brezis [21, Prop. 3.12]),

$$\|u-u_0\|_{[L^\infty((0,t);W^{-1,1}(I))]^M}\leq \liminf \|u^\varepsilon-u_0^\varepsilon\|_{[L^\infty((0,t);W^{-1,1}(I))]^M}\leq Ct^{\frac{1}{2}}.$$

From (6.38) we deduce that

$$\|u(t)-u_0\|_{[W^{-1,1}(I)]^M}\leq Ct^{\frac{1}{2}},$$

which proves that $u(\cdot,0)=u_0$ in $[\mathcal{D}'(\mathbb{R})]^M$.

Step 3 (Continuity of solution) :
Now, we are going to prove the continuity estimate (1.5). For all $h>0$ and $(t,x)\in(0,T)\times\mathbb{R}$, we have :

$$
\begin{aligned}
|u(t,x+h)-u(t,x)| &\leq \left|\int_x^{x+h} \partial_x u(t,y)dy\right| \\
&\leq 2\|1\|_{EXP(x,x+h)}\|\partial_x u\|_{L\log L(x,x+h)}, \\
&\leq 2\frac{1}{\ln(\frac{1}{h}+1)}\|\partial_x u\|_{L^\infty((0,T);L\log L(\mathbb{R}))}, \\
&\leq C\frac{1}{\ln(\frac{1}{h}+1)},
\end{aligned}
$$

where we have used in the second line the generalized Hölder inequality (see Lemma 6.4) and in last line we have used that $\partial_x u \in L^\infty((0,T);L\log L(\mathbb{R}))$. Which proves finally the continuity in space. Now, we prove the continuity in time, for all $\delta > 0$ and $(t,x) \in (0,T)\times\mathbb{R}$, we have :

$$
\begin{aligned}
\delta|u(t+\delta,x)-u(t,x)| &= \int_x^{x+\delta} |u(t+\delta,x)-u(t,x)|dy, \\
&\leq \overbrace{\int_x^{x+\delta} |u(t+\delta,x)-u(t+\delta,y)|dy}^{K_1}, \\
&\quad + \overbrace{\int_x^{x+\delta} |u(t+\delta,y)-u(t,y)|dy}^{K_2}, \\
&\quad + \overbrace{\int_x^{x+\delta} |u(t,y)-u(t,x)|dy}^{K_3}.
\end{aligned}
$$

Similarly, as in the last estimate, we can show that :

$$
\begin{aligned}
K_1+K_3 &\leq \delta\int_x^{x+\delta} |\partial_x u(t+\delta,y)|dy, +\delta\int_x^{x+\delta} |\partial_x u(t,y)|dy, \\
&\leq 4\delta\|1\|_{EXP(x,x+\delta)}\|\partial_x u\|_{L^\infty((0,T);L\log L(\mathbb{R}))}, \\
&\leq C\frac{\delta}{\ln(\frac{1}{\delta}+1)}.
\end{aligned}
$$

Now, we use that u is a solution of (P), and we obtain that :

$$\begin{aligned}
K_2 &\leq \int_x^{x+\delta}\int_t^{t+\delta} |\partial_t u(s,y)|dy,\\
&\leq \int_t^{t+\delta}\int_x^{x+\delta} |a(u(s,y))\diamond\partial_x u(s,y)|dsdy,\\
&\leq \delta M_0 \|u\|_{L^\infty((0,T)\times\mathbb{R})}\|1\|_{EXP(x,x+\delta)}\|\partial_x u\|_{L^\infty((0,T);L\log L(\mathbb{R})},\\
&\leq C\frac{\delta}{\ln(\frac{1}{\delta}+1)},
\end{aligned}$$

where we have used in last line that $u \in L^\infty((0,T)\times\mathbb{R})$, collecting the estimates of K_1, K_2 and K_3, we prove that :

$$|u(t+\delta,x)-u(t,x)| \leq \frac{1}{\delta}(K_1+K_2+K_3) \leq C\frac{1}{\ln(\frac{1}{\delta}+1)},$$

which proves finally the following :

$$|u(t+\delta,x+h)-u(t,x)| \leq C\left(\frac{1}{\ln(\frac{1}{\delta}+1)}+\frac{1}{\ln(\frac{1}{h}+1)}\right).$$

□

7 Some remarks on the uniqueness

In this Section we study the uniqueness of solution of the system (P)-(ID) with

$$a^i(u) = \sum_{j=1,\dots,M} A_{ij}u^j.$$

We show some uniqueness results for some particular matrices with $M \geq 2$.
For the proof of Theorem 1.5 in Subsection 7.2, we need to recall in the following Subsection the definition of viscosity solution and some well-known results in this framework.

7.1 Some useful results for viscosity solutions

The notion of viscosity solutions is quite recente. This concept has been introduced by Crandall and Lions [38, 39] in 1980, to solve the first-order Hamilton-Jacobi equations. The theory then extended to the second order equations by the work of Jensen [88] and Ishii [81]. For good introduction of this theory, we refer to Barles [12] and Bardi, Capuzzo-Dolcetta [10].

Now, we recall the definition of viscosity solution for the following problem for all $0 \leq \varepsilon \leq 1$:

$$\partial_t v + H(t, x, v, \partial_x v) - \varepsilon \partial_{xx} v = 0 \quad \text{with} \quad x, v \in \mathbb{R},\ t \in (0, T). \tag{7.39}$$

where $H : (0, T) \times \mathbb{R}^3 \longmapsto \mathbb{R}$ is the Hamiltonian and is supposed continuous. We will set

$$USC((0, T) \times \mathbb{R}) = \{f \text{ such that } f \text{ is upper semicontinuous on } (0, T) \times \mathbb{R}\},$$

$$LSC((0, T) \times \mathbb{R}) = \{f \text{ such that } f \text{ is lower semicontinuous on } (0, T) \times \mathbb{R}\}.$$

Definition 7.1 (Viscosity subsolution, supersolution and solution)
A function $v \in USC((0, T) \times \mathbb{R})$ is a viscosity subsolution of (7.39) if it satisfies, for every $(t_0, x_0) \in (0, T) \times \mathbb{R}$ and for every test function $\phi \in C^2((0, T) \times \mathbb{R})$, that is tangent from above to v at (t_0, x_0), the following holds :

$$\partial_t \phi + H(t_0, x_0, v, \partial_x \phi) - \varepsilon \partial_{xx} \phi \leq 0.$$

A function $v \in LSC((0, T) \times \mathbb{R})$ is a viscosity supersolution of (7.39) if it satisfies, for every $(t_0, x_0) \in (0, T) \times \mathbb{R}$ and for every test function $\phi \in C^2((0, T) \times \mathbb{R})$, that is tangent from below to v at (t_0, x_0), the following holds :

$$\partial_t \phi + H(t_0, x_0, v, \partial_x \phi) - \varepsilon \partial_{xx} \phi \geq 0.$$

A function v is a viscosity solution of (7.39) if, and only if, it is a sub and a supersolution of (7.39).

Let us now recall some well-known results.

Remark 7.2 (Classical solution-viscosity solution)
If v is a C^2 solution of (7.39), then v is a viscosity solution of (7.39).

***Lemma* 7.3 (Stability result**, *see Barles [12, Th 2.3]*)
We suppose that, for $\varepsilon > 0$, v^ε is a viscosity solution of (7.39). If $v^\varepsilon \to v$ uniformly on every compact set then v is a viscosity solution of (7.39) with $\varepsilon = 0$.

***Lemma* 7.4 (Gronwall for viscosity solution)**
Let v, a locally bounded $USC(0, T)$ function, which is a viscosity subsolution of the equation $\dfrac{d}{dt} v = \alpha v$ where $\alpha \geq 0$. Assume that $v(0) \leq v_0$ then $v \leq v_0\, e^{\alpha T}$ in $(0, T)$.

The proof of this Lemma is a direct application of the comparison principle, (see Barles [12, Th 2.4]).

Remark 7.5
From Lemmata 7.2, 7.3 and from (6.35), we can notice that the solution u^i of our system (P) given in Theorem 1.1 is also a viscosity solution of (P) (where the u^j for $j \neq i$ are considered fixed to apply Definition 7.1).

7.2 Uniqueness results

In this Subsection we prove Theorem 1.5. Before going on, we recall in the following Remark a well-known uniqueness results and we recall in Theorem 7.7 the uniqueness results of $W^{1,\infty}$ solution of (P).

Remark 7.6 (Uniqueness for quasi-monotone Hamiltonians)
If the elements of the matrix A satisfy :

$$A_{ii} + \sum_{j \neq i, A_{ij} < 0} A_{ij} \geq 0 \quad \textit{for all} \quad i = 1, \cdots, M.$$

and if $\partial_x u^i \geq 0$ for $i = 1, \ldots, M$, then we can easily check that the Hamiltonian

$$H_i(u, \partial_x u^i) = \left(\sum_{j=1,\ldots M} A_{ij} u^j \right) \partial_x u^i,$$

is quasi-monotone in the sense of Ishii, Koike [83, (A.3)]. Then the result of Ishii, Koike [83, Th.4.7] shows that for any initial condition $u_0 \in [L^\infty(\mathbb{R})]^M$ satisfying $(H1)$-$(H2)$, the system (P) satisfies the comparison principle which implies the uniqueness of the solution.

We have the following result which seems quite standard :

Theorem 7.7 (Uniqueness of the $W^{1,\infty}$ solution)
Let $u_0 \in [W^{1,\infty}(\mathbb{R})]^M$ and $T > 0$. Then system (P)-(ID) admits a unique solution in $[W^{1,\infty}([0,T) \times \mathbb{R})]^M$.

The proof of this Theorem is given in Appendix, because we have not found any proof of such a result in the literature.

Proof of Theorem 1.5 :
Using Theorem 7.7 with $a^i(u) = \sum_{j=1,\ldots,M} A_{ij} u^j$, it is enough to show that the system (P)-(ID) admits a solution in $[W^{1,\infty}([0,T) \times \mathbb{R})]^M$. To do that, it is enough to prove that the solution u^ε of the approximated system obtained in Corollary 5.5 satisfies that $\partial_x u^\varepsilon$ is bounded in $[L^\infty((0,T) \times \mathbb{R})]^M$ uniformly in $0 < \varepsilon \leq 1$. Indeed, we then

get the same property for $\partial_x u$, where u is the limit of u^ε as $\varepsilon \to 0$. Moreover, from the equation (P) satisfied by u and the fact that

$$u \in [L^\infty((0,T)\times\mathbb{R})]^M \quad \text{and} \quad \partial_x u \in [L^\infty((0,T)\times\mathbb{R})]^M,$$

we deduce that $\partial_t u \in [L^\infty((0,T)\times\mathbb{R})]^M$ which shows that $u \in [W^{1,\infty}([0,T)\times\mathbb{R})]^M$.

To simplify, we denote

$$w^\varepsilon = \partial_x u^\varepsilon,$$

and we interest in the

$$\max_{x\in\mathbb{R}} w^{\varepsilon,i}(t,x) = m_i(t).$$

This maximum is reached at least at some point $x_i(t)$, because $w^{\varepsilon,i} \in C^\infty((0,T)\times\mathbb{R}) \cap W^{1,p}((0,T)\times\mathbb{R})$ for all $1 < p \leq +\infty$ (see Lemma 4.1, (4.19)).

In the following we prove in the two cases (i) and (ii) defined in Theorem 1.5 that m_i, for all $i = 1,\dots,M$, is bounded uniformly in ε. First, deriving with respect to x the equation (P_ε) satisfied by $u^\varepsilon \in [C^\infty((0,T)\times\mathbb{R})]^M$, we can see that w^ε satisfies the following equation

$$\partial_t w^{\varepsilon,i} - \varepsilon\partial_{xx} w^{\varepsilon,i} + \sum_{j=1,\dots,M} A_{ij} u^{\varepsilon,j}\partial_x w^{\varepsilon,i} + \sum_{j=1,\dots,M} A_{ij} w^{\varepsilon,j} w^{\varepsilon,i} = 0. \tag{7.40}$$

Now, we prove that m_i is a viscosity subsolution of the following equation,

$$\frac{d}{dt} m_i(t) + \sum_{j=1,\dots,M} A_{ij} w^{\varepsilon,j}(t,x_i(t)) w^{\varepsilon,i}(t,x_i(t)) \leq 0. \tag{7.41}$$

Indeed, let $\phi \in C^2(0,T)$ a test function, such that $\phi \geq m_i$ and $\phi(t_0) = m_i(t_0)$ for some $t_0 \in (0,T)$. From the definition of m_i, we can easily check that $\phi \geq w^{\varepsilon,i}(t,x)$ and $\phi(t_0) = w^{\varepsilon,i}(t_0,x_i(t_0))$. But, the fact that $w^{\varepsilon,i} \in C^\infty((0,T)\times\mathbb{R})$, by Remark 7.2 we know that $w^{\varepsilon,i}$ is a viscosity subsolution of (7.40). We apply Definition 7.1, and the fact that $\partial_x\phi = \partial_{xx}\phi = 0$, we get

$$\frac{d}{dt}\phi(t_0) + \sum_{j=1,\dots,M} A_{ij} w^{\varepsilon,j}(t_0,x_i(t_0)) w^{\varepsilon,i}(t_0,x_i(t_0)) \leq 0.$$

Which proves that m_i is a viscosity subsolution of (7.41).

Two cases may accur :

i) Here, we consider the case where $M \geq 2$ and $A_{ij} \geq 0$ for all $j \geq i$. We see the equation satisfied by m_1, we deduce that satisfies (a viscosity subsolution)

$$\frac{d}{dt}m_1(t) \leq -\sum_{j=1,\dots,M} A_{1j}w^{\varepsilon,j}(t,x_1(t))w^{\varepsilon,1}(t,x_1(t)) \leq 0,$$

where we have used the fact that, for $j = 1,\dots,M$, $A_{1j} \geq 0$ and $w^{\varepsilon,j} \geq 0$. This proves by Lemma 7.4 (with $\alpha = 0$) that,

$$m_1(t) \leq m_1(0) = w^{\varepsilon,1}(t,x_1(t)) \leq \|\partial_x u_0^1\|_{L^\infty(\mathbb{R})}.$$

We reason by recurrence : we assume that $m_j \leq C$ for all $j \leq i$, where C is a positive constant independent of ε, and we prove that m_{i+1} is bounded uniformly in ε. Indeed, we know that

$$\begin{aligned}
\frac{d}{dt}m_{i+1}(t) &\leq -\sum_{j=1,\dots,M} A_{i+1,j}w^{\varepsilon,j}(t,x_j(t))w^{\varepsilon,i+1}(t,x_{i+1}(t)),\\
&\leq -\sum_{j<i+1} A_{i+1,j}w^{\varepsilon,j}(t,x_j(t))w^{\varepsilon,i+1}(t,x_{i+1}(t))\\
&\quad -\sum_{M\geq j\geq i+1} A_{i+1,j}w^{\varepsilon,j}(t,x_j(t))w^{\varepsilon,i+1}(t,x_{i+1}(t)),
\end{aligned}$$

We use that $A_{i+1,j} \geq 0$, for $M \geq j \geq i+1$, we obtain that

$$\begin{aligned}
\frac{d}{dt}m_{i+1}(t) &\leq -\sum_{j<i+1} A_{i+1,j}w^{\varepsilon,j}(t,x_j(t))w^{\varepsilon,i+1}(t,x_{i+1}(t)),\\
&\leq C\left(\sum_{j<i+1}|A_{i+1,j}|\right)m_{i+1}(t).
\end{aligned}$$

This implies by Lemma 7.4, with $\alpha = C\left(\sum_{j<i+1}|A_{i+1,j}|\right)$, that

$$\begin{aligned}
m_{i+1}(t) &\leq m_{i+1}(0)e^{\alpha T},\\
&\leq \|\partial_x u_0^{i+1}\|_{L^\infty(\mathbb{R})}e^{\alpha T}.
\end{aligned}$$

Which proves that for all $i = 1,\dots,M$, m_i is bounded uniformly in ε.

ii) Here, we consider the case where $M \geq 2$ and $A_{ij} \leq 0$ for all $i \neq j$. Taking the sum over the index i, from (7.41) we get that the quantity $m(t) = \sum_{i=1,\dots,M} m_i(t)$ satisfies (a viscosity subsolution see Bardi et al. [11])

$$\begin{aligned}\frac{d}{dt}m(t) &\leq -\sum_{i,j=1,\dots,M} A_{ij}w^{\varepsilon,j}(t,x_i(t))w^{\varepsilon,i}(t,x_i(t)),\\ &\leq -\sum_{i,j=1,\dots,M} A_{ij}w^{\varepsilon,j}(t,x_j(t))w^{\varepsilon,i}(t,x_i(t)),\\ &\leq 0.\end{aligned}$$

where we have used that the matrix A satisfies $(H2')$ and $w^{\varepsilon,i} \geq 0$, for $i = 1, \dots, M$. Using Lemma 7.4 with $\alpha = 0$, we get

$$\begin{aligned}m(t) &\leq m(0) = \sum_{i=1,\dots,M} \partial_x u_0^{\varepsilon,i},\\ &\leq \sup_{y\in\mathbb{R}} \sum_{i=1,\dots,M} \partial_x u_0^i(y).\end{aligned}$$

which proves (1.7). □

8 Application on the dynamics of dislocations densities

In this Section, we present a model describing the dynamics of dislocations densities. We refer to [74] for a physical presentation of dislocations which are (moving) defects in crystals. Even if the problem is naturally a three-dimensional problem,we will first assume that the geometry of the problem is invariant by translations in the x_3-direction. This reduces the problem to the study of dislocations densities defined on the plane (x_1, x_2) and propagation in a given direction $\vec{b}$ belonging to the plane (x_1, x_2) (which is called the "Burger's vector").

In this setting we consider a finite number of slip directions $\vec{b} \in \mathbb{R}^2$ and to each $\vec{b}$ we will associate a dislocation density. For a detailed physical presentation of a model with multi-slip directions, we refer to Yefimov, Van der Giessen [136] and Yefimov [135, ch. 5.] and to Groma, Balogh [71] for the case of a model with a single slip direction . See also Cannone et al. [25] for a mathematical analysis of the Groma, Balogh model. In Subsection **??**, we present the 2D-model with multi-slip directions.

In the particular geometry where the dislocations densities only depend on the variable $x = x_1 + x_2$, this two-dimensional model reduces to one-dimensional model which presented in In Subsection 8.2. See El Hajj [47] and El Hajj, Forcadel [48] for a study in the special case of a single slip direction. Finally in Subsection 8.3,

we explain how to recover equation (P) as a model for dislocation dynamics with $a^i(u) = \sum_{j=1,\dots,M} A_{ij}u^j$ for some particular non-negative and symmetric matrix A.

8.1 The 2D-model

We now present in details the two-dimensional model. We denote by $\mathbf{X}$ the vector $\mathbf{X} = (x_1, x_2)$. We consider a crystal filling the whole space $\mathbb{R}^2$ and its displacement $v = (v_1, v_2) : \mathbb{R}^2 \to \mathbb{R}^2$, where we have not yet introduced the time dependence for the moment.

We define the total strain by

$$\varepsilon(v) = \frac{1}{2}(\nabla v + {}^t\nabla v),$$

where ∇v is the gradient with $(\nabla v)_{ij} = \dfrac{\partial v_i}{\partial x_j}, \quad i, j \in \{1, 2\}$.

Now, we assume that the dislocations densities under consideration are associated to edge dislocations. This means that we consider M slip directions where each direction is caraterize by a Burgers vectors $\vec{b}^k = (b_1^k, b_2^k) \in \mathbb{R}^2$, for $k = 1, \dots, M$. This leads to M type of dislocations which propagate in the plan (x_1, x_2) following the direction of $\vec{b}^k$, for $k = 1, \dots, M$.

The total strain can be splitted in two parts :

$$\varepsilon(v) = \varepsilon^e + \varepsilon^p.$$

Here, ε^e is the elastic strain and ε^p the plastic strain defined by

$$\varepsilon^p = \sum_{k=1,\dots,M} \varepsilon^{0,k} u^k, \tag{8.42}$$

where, for each $k = 1, \dots, M$, the scalar function u^k is the plastic displacement associated to the k-th slip system whose matrix $\varepsilon^{0,k}$ is defined by

$$\varepsilon^{0,k} = \frac{1}{2}\left(\vec{b}^k \otimes \vec{n}^k + \vec{n}^k \otimes \vec{b}^k\right), \tag{8.43}$$

where $\vec{n}^k$ is unit vector orthogonal to $\vec{b}^k$ and $\left(\vec{b}^k \otimes \vec{n}^k\right)_{ij} = b_i^k n_j^k$.

To simplify the presentation, we assume the simplest possible periodicity property of the unknowns.

Assumption (H) :

i) The function v is $\mathbb{Z}^2$-periodic with $\displaystyle\int_{(0,1)^2} v\, d\mathbf{X} = 0$.

ii) For each $k = 1, \ldots, M$, there exists $L^k \in \mathbb{R}^2$ such that $u^k - L^k \cdot \mathbf{X}$ is a $\mathbb{Z}^2$-periodic.

iii) The integer M is even with $M = 2N$ and $L^{k+N} = L^k$, and that

$$L^{k+N} = L^k, \quad \vec{b}^{k+N} = -\vec{b}^k, \quad \vec{n}^{k+N} = \vec{n}^k,$$
$$\varepsilon^{0,k+N} = -\varepsilon^{0,k}.$$

iv) We denote by $\vec{\tau}^k = (\tau_1^k, \tau_2^k)$ a vector parallel to $\vec{b}^k$ such that $\vec{\tau}^{k+N} = \vec{\tau}^k$. We require that L^k is chosen such $\vec{\tau}^k \cdot L^k \geq 0$.

The plastic displacement u^k is related to the dislocation density associated to the Burgers vector $\vec{b}^k$. We have

$$k\text{-th dislocation density } = \vec{\tau}^k \cdot \nabla u^k \geq 0. \tag{8.44}$$

The stress is then given by

$$\sigma = \Lambda : \varepsilon^e, \tag{8.45}$$

i.e. the coefficients of the matrix σ are :

$$\sigma_{ij} = \sum_{k,l=1,2} \Lambda_{ijkl} \varepsilon^e_{kl} \quad \text{for} \quad i,j = 1,2,$$

where $\Lambda = (\Lambda_{ijkl})_{i,j,k,l=1,2}$, are the constant elastic coefficients of the material, satisfying for $m > 0$:

$$\sum_{ijkl=1,2} \Lambda_{i,j,k,l} \varepsilon_{ij} \varepsilon_{kl} \geq m \sum_{i,j=1,2} \varepsilon_{ij}^2 \tag{8.46}$$

for all symmetric matrices $\varepsilon = (\varepsilon_{ij})_{ij}$, *i.e.* such that $\varepsilon_{ij} = \varepsilon_{ji}$.

Finally, for $k = 1, \ldots, M$, the functions u^k and v are then assumed to depend on $(t, \mathbf{X}) \in (0,T) \times \mathbb{R}^2$ and to be solutions of the coupled system (see Yefimov [135, ch. 5.] and Yefimov, Van der Giessen [136]) :

$$\begin{cases} \operatorname{div} \sigma &= 0 & \text{on } (0,T) \times \mathbb{R}^2, \\ \sigma &= \Lambda : (\varepsilon(v) - \varepsilon^p) & \text{on } (0,T) \times \mathbb{R}^2, \\ \varepsilon(v) &= \frac{1}{2}(\nabla v + {}^t\nabla v) & \text{on } (0,T) \times \mathbb{R}^2, \\ \varepsilon^p &= \displaystyle\sum_{k=1,\ldots,M} \varepsilon^{0,k} u^k & \text{on } (0,T) \times \mathbb{R}^2, \\ \\ \partial_t u^k &= (\sigma : \varepsilon^{0,k}) \vec{\tau}^k . \nabla u^k & \text{on } (0,T) \times \mathbb{R}^2, \quad \text{for } k = 1, \ldots, M, \end{cases} \tag{8.47}$$

i.e. in coordinates

$$\left\{\begin{array}{lll} \displaystyle\sum_{j=1,2} \frac{\partial \sigma_{ij}}{\partial x_j} = 0 & \text{on } (0,T)\times\mathbb{R}^2, & \text{for } i=1,2, \\ \left.\begin{array}{ll} \sigma_{ij} = \displaystyle\sum_{k,l=1,2} \Lambda_{ijkl}\left(\varepsilon_{kl}(v) - \varepsilon^p_{kl}\right) & \text{on } (0,T)\times\mathbb{R}^2, \\ \varepsilon_{ij}(v) = \dfrac{1}{2}\left(\dfrac{\partial v_i}{\partial x_j} + \dfrac{\partial v_j}{\partial x_i}\right) & \text{on } (0,T)\times\mathbb{R}^2, \\ \varepsilon^p_{ij} = \displaystyle\sum_{k=1,\dots,M} \varepsilon^{0,k}_{ij} u^k & \text{on } (0,T)\times\mathbb{R}^2, \end{array}\right| & \text{for } i,j=1,2 \\ \partial_t u^k = \left(\displaystyle\sum_{i,j\in\{1,2\}} \sigma_{ij}\varepsilon^{0,k}_{ij}\right) \vec{\tau}^k.\nabla u^k & \text{on } (0,T)\times\mathbb{R}^2, & \text{for } k=1,\dots,M, \end{array}\right. \tag{8.48}$$

where the unknowns of the system are u^k and the displacement $v = (v_1, v_2)$ and with $\varepsilon^{0,k}$ defined in (8.43). Here the first equation of (8.47) is the equation of elasticity, while the last equation of (8.47) is the transport equation satisfied by the plastic displacement whose velocity is given by the Peach-Koehler force $\sigma : \varepsilon^{0,k}$. Remark that this implies in particular that each dislocation density satisfies a conservation law (see the equation obtained by derivation, using (8.44)). Remark also that our equations are compatible with our periodicity assumptions (H), (i)-(ii).

8.2 Derivation of the 1D-model

In this Subsection we are interested in a particular geometry where the dislocations densities depend only on the variable $x = x_1 + x_2$. This will lead to 1D-model. More precisely, we make the following :

Assumption (H') :
i) The functions $v(t,\mathbf{X})$ and $u^k(t,\mathbf{X}) - L^k \cdot \mathbf{X}$ depend on the variable $x = x_1 + x_2$.

ii) $\tau^k_1 + \tau^k_2 = 1$, for $k = 1,\dots,M$.

iii) $L^k_1 = L^k_2$ for $k = 1,\dots,M$.

For this particular one-dimensional geometry, we denote by an abuse of notation the function $v = v(t,x)$ which is 1-periodic in x. If we set $l^k = \frac{L_1^k + L_2^k}{2}$, we have

$$L^k \cdot \mathbf{X} = l^k \cdot x + \left(\frac{L_1^k - L_2^k}{2}\right)(x_1 - x_2).$$

By assumption (H'), (iii), we see (again by an abuse of notation) that $u = (u^k(t,x))_{k=1,\dots,M}$ is such that for $k = 1,\dots,M$, $u^k(t,x) - l^k \cdot x$ is 1-periodic in x.

Now, we can integrate the equations of elasticity, *i.e.* the first equation of (8.47). Using the $\mathbb{Z}^2$-periodicity of the unknowns (see assumption (H), (i)-(ii)), and the fact that $\varepsilon^{0,k+N} = -\varepsilon^{0,k}$ (see assumption (H), (iii)), we can easily conclude that the strain

$$\varepsilon^e \text{ as a linear function of } (u^j - u^{j+N})_{j=1,\dots,N} \quad \text{and of } \left(\int_0^1 (u^j - u^{j+N})\, dx\right)_{j=1,\dots,N}. \tag{8.49}$$

This leads to the following Lemma

***Lemma* 8.1 (Stress for the 1D-model)**
Under assumptions (H), (i)-(ii)-(iii) and (H'), (i)-(iii) and (8.46), we have

$$-\sigma : \varepsilon^{0,i} = \sum_{j=1,\dots,M} A_{ij} u^j + \sum_{j=1,\dots,M} Q_{ij} \int_0^1 u^j\, dx, \quad \textit{for } i = 1,\dots,N. \tag{8.50}$$

where for $i,j = 1,\dots,N$

$$\begin{cases} A_{i,j} = A_{j,i} \quad \textit{and} \quad A_{i+N,j} = -A_{i,j} = A_{i,j+N}, \\ Q_{i,j} = Q_{j,i} \quad \textit{and} \quad Q_{i+N,j} = -Q_{i,j} = Q_{i,j+N}. \end{cases} \tag{8.51}$$

Moreover the matrix A is non-negative.

The proof of Lemma 8.1 will be given at the end of this Subsection.

Finally using Lemma 8.1, we can eliminate the stress and reduce the problem to a one-dimensional system of M transport equations only depending on the function u^i, for $i = 1,\dots,M$. Naturally, from (8.50) and (H'), (ii) this 1D-model has the following form

$$\partial_t u^i + \left(\sum_{j=1,\dots,M} A_{ij} u^j + \sum_{j=1,\dots,M} Q_{ij} \int_0^1 u^j \, dx \right) \partial_x u^i = 0, \qquad \text{on } (0,T) \times \mathbb{R}, \quad \text{for } i = 1, \dots, M, \tag{8.52}$$

with from (8.44)

$$\partial_x u^i \geq 0 \quad \text{for } i = 1, \dots, M. \tag{8.53}$$

Now, we give the proof of Lemma 8.1.

Proof of Lemma 8.1 :
For the 2D-model, let us consider the elastic energy on the periodic cell (using the fact that ε^e is $\mathbb{Z}^2$-periodic)

$$E^{el} = \frac{1}{2} \int_{(0,1)^2} \Lambda : \varepsilon^e : \varepsilon^e \, d\mathbf{X}.$$

By definition of σ and ε^e, we have for $i = 1, \dots, M$

$$\sigma : \varepsilon^{0,i} = -\nabla_{u^i} E^{el}. \tag{8.54}$$

On the other hand usind (H'), (i)-(iii), (with $x = x_1 + x_2$) we can check that we can rewrite the elastic energy as

$$E^{el} = \frac{1}{2} \int_0^1 \Lambda : \varepsilon^e : \varepsilon^e \, dx.$$

Replacing ε^e by its expression (8.49), we get :

$$\begin{aligned} E^{el} = \; & \frac{1}{2} \int_0^1 \sum_{i,j=1,\dots,N} A_{ij} (u^j - u^{j+N})(u^i - u^{i+N}) \, dx \\ & + \frac{1}{2} \sum_{i,j=1,\dots,N} Q_{ij} \left(\int_0^1 (u^j - u^{j+N}) \, dx \right) \left(\int_0^1 (u^i - u^{i+N}) \, dx \right), \end{aligned}$$

for some symmetric matrices $A_{i,j} = A_{j,i}$, $Q_{i,j} = Q_{j,i}$. In particular, joint to (8.54) this gives exactly (8.50) with (8.51).

Let us now consider the functions $w^i = u^i - u^{i+N}$ such that

$$\int_0^1 w^i \, dx = 0 \quad \text{for i=1,...,N}, \tag{8.55}$$

From (8.46) that we deduce that

$$0 \leq E^{el} = \frac{1}{2}\int_0^1 \sum_{i,j=1,\dots,N} A_{ij} w^i w^j \; dx.$$

More precisely, for all $i = 1, \dots, N$ and for all $\bar{w}^i \in \mathbb{R}$, we set

$$w^i = \begin{cases} \bar{w}^i & \text{on} \quad [0, \frac{1}{2}], \\ -\bar{w}^i & \text{on} \quad [\frac{1}{2}, 1], \end{cases}$$

which satisfies (8.55). Finally, we obtain that

$$0 \leq E^{el} = \frac{1}{2}\int_0^1 \sum_{i,j=1,\dots,N} A_{ij} \bar{w}^i \bar{w}^j \; dx.$$

Because this is true for every $\bar{w}^i$, we deduce that A a non-negative matrix. □

8.3 Heuristic derivation of the non-periodic model

Starting from the model (8.52)-(8.53) where for $i = 1, \dots, M,$, $u^i(t,x) - l^i \cdot x$ is 1-periodic in x, we now want to rescale the unknowns to make the periodicity disappear. More precisely, we have the following Lemma :

***Lemma* 8.2 (Non-periodic model)**
Let u be a solution of (8.52)-(8.53) assuming Lemma 8.1 and $u^i(t,x) - l^i \cdot x$ is 1-periodic in x. Let

$$u^j_\delta(t,x) = u^j(\delta t, \delta x), \quad \textit{for a small } \delta > 0 \textit{ and for } j = 1, \dots, M,$$

such that, for all $j = 1, \dots, M$

$$u^j_\delta(0,\cdot) \to \bar{u}^j(0,\cdot), \quad \textit{as} \quad \delta \to 0, \quad \textit{and} \quad \bar{u}^j(0,\pm\infty) = \bar{u}^{j+N}(0,\pm\infty) \tag{8.56}$$

Then $\bar{u} = (\bar{u}^j)_{j=1,\dots,M}$ formally is a solution of

$$\partial_t \bar{u}^i + \left(\sum_{j=1,\dots,M} A_{ij} \bar{u}^j \right) \partial_x \bar{u}^i = 0, \qquad \textit{on } (0,T) \times \mathbb{R}, \tag{8.57}$$

with the matrix A is non-negative and $\partial_x \bar{u}^i \geq 0$ for $i = 1, \dots, M$.

We remark that the limit problem (8.57) is of type (P) with $(H1')$ and $(H2')$.

Now, we give a formal proof of Lemma 8.2.

Formal proof of Lemma 8.2 :
Here, we know that $u_\delta^i - \delta l^i \cdot x$ is $\frac{1}{\delta}$-periodic in x, and satisfies for $i = 1, \dots, M$

$$\partial_t u_\delta^i + \left(\sum_{j=1,\dots,M} A_{ij} u_\delta^j + \delta \sum_{j=1,\dots,M} Q_{ij} \int_0^{\frac{1}{\delta}} u_\delta^j \, dx \right) \partial_x u_\delta^i = 0, \qquad \text{on } (0,T) \times \mathbb{R}, \tag{8.58}$$

To simplify, assume that the initial data $u_\delta(0,\cdot)$ converge to a function $\bar{u}(0,\cdot)$ such that $\partial_x u_\delta(0,\cdot)$ has a support in $(-R,R)$, uniformly in δ, where R a positve constant. We expect heuristically that the velocity in (8.58) remains uniformly bounded as $\delta \to 0$.

Therefore, using the finite propagation speed, we see that, there exists a constant C independent in δ, such that $\partial_x u_\delta(t,\cdot)$ has a support in $(-R-Ct, R+Ct)$ uniformly in δ. Moreover, from (8.56) and the fact that

$$\sum_{j=1,\dots,M} Q_{ij} \int_0^{\frac{1}{\delta}} u_\delta^j \, dx = \sum_{j=1,\dots,N} Q_{ij} \int_0^{\frac{1}{\delta}} (u^j - u^{j+N}) \, dx,$$

we deduce that

$$\sum_{j=1,\dots,M} Q_{ij} \int_0^{\frac{1}{\delta}} u_\delta^j \, dx,$$

remains bounded uniformly in δ. Then formally the non-local term vanishes and we get for $i = 1, \dots, M$

$$\sum_{j=1,\dots,M} A_{ij} u_\delta^j + \delta \sum_{j=1,\dots,M} Q_{ij} \int_0^{\frac{1}{\delta}} u_\delta^j \, dx \to \sum_{j=1,\dots,M} A_{ij} \bar{u}^j, \quad \text{as} \quad \delta \to 0,$$

which proves that $\bar{u}$ is solution of (8.57), with the matrix A is non-negative . □

9 Appendix : proof of Theorem 7.7

Let $u_1 = (u_1^i)_i$ and $u_2 = (u_2^i)_i$, for $i = 1, \cdots, M$, be two solutions of the system (P) in $[W^{1,\infty}((0,T) \times \mathbb{R})]^M$, such that $u_1^i(0,\cdot) = u_2^i(0,\cdot)$.

Then by definition u_1^i and u_2^i satisfy respectively the following system, for $i = 1, \cdots, M$:

$$\partial_t u_1^i = -a^i(u_1)\partial_x u_1^i,$$

$$\partial_t u_2^i = -a^i(u_2)\partial_x u_2^i,$$

Subtracting the two equations we get :

$$\partial_t \left(u_1^i - u_2^i\right) = -\left(a^i(u_1) - a^i(u_2)\right)\partial_x u_1^i - a^i(u_2)\partial_x(u_1^i - u_2^i).$$

Multiplying this system by $(u_1^i - u_2^i)(\psi)^2$ where $\psi(x) = e^{-|x|}$, and integrating in space, we deduce that :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|(u_1^i - u_2^i)\psi\right\|^2_{L^2(\mathbb{R})} = & -\int_{\mathbb{R}}\left(a^i(u_1) - a^i(u_2)\right)\left(u_1^i - u_2^i\right)\psi^2\partial_x u_1^i \\ & -\int_{\mathbb{R}} a^i(u_2)\psi^2\left(u_1^i - u_2^i\right)\partial_x(u_1^i - u_2^i).\end{aligned}$$

Taking the sum over i, we get :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\sum_{i=1,\dots,M}\left\|(u_1^i - u_2^i)\psi\right\|^2_{L^2(\mathbb{R})}\right) = & \overbrace{-\int_{\mathbb{R}}\sum_{i=1,\dots,M}\left(a^i(u_1) - a^i(u_2)\right)\left(u_1^i - u_2^i\right)\psi^2\partial_x u_1^i}^{I_1} \\ & \overbrace{-\frac{1}{2}\int_{\mathbb{R}}\sum_{i=1,\dots,M} a^i(u_2)\psi^2\partial_x(u_1^i - u_2^i)^2}^{I_2}.\end{aligned}$$

Integrating I_2 by part, we obtain :

$$\begin{aligned}I_2 = & \overbrace{\frac{1}{2}\int_{\mathbb{R}}\sum_{i,j=1,\dots,M} a^i_{,j}(u_2)(\partial_x u_2^j)\psi^2(u_1^i - u_2^i)^2}^{I_{21}} \\ & + \overbrace{\frac{1}{2}\int_{\mathbb{R}}\sum_{i=1,\dots,M} a^i(u_2)(u_1^i - u_2^i)^2\partial_x(\psi^2)}^{I_{22}}.\end{aligned}$$

Next, using the fact that u_2^i is bounded in $W^{1,\infty}((0,T)\times\mathbb{R})$, for $i = 1, \dots, M$, we deduce that :

$$
\begin{aligned}
|I_{21}| &\leq \tfrac{1}{2} M M_1 \|u_2\|_{[W^\infty((0,T)\times\mathbb{R})]^M} \left(\sum_{i=1,\dots,M} \left\| (u_1^i - u_2^i)\psi \right\|^2_{L^2(\mathbb{R})} \right), \\
&\leq C \left(\sum_{i=1,\dots,M} \left\| (u_1^i - u_2^i)\psi \right\|^2_{L^2(\mathbb{R})} \right).
\end{aligned} \tag{9.59}
$$

Since $\partial_x(\psi(x))^2 = -2sign(x)(\psi(x))^2$ and u_2^i is bounded in $W^{1,\infty}((0,T)\times\mathbb{R})$, for $i = 1, \cdots, M$, we obtain :

$$
\begin{aligned}
|I_{22}| &\leq \tfrac{1}{2} M_0 \left(\sum_{i=1,\dots,M} \left\| (u_1^i - u_2^i)\psi \right\|^2_{L^2(\mathbb{R})} \right) \\
&\leq C \left(\sum_{i=1,\dots,M} \left\| (u_1^i - u_2^i)\psi \right\|^2_{L^2(\mathbb{R})} \right)
\end{aligned} \tag{9.60}
$$

Now, using the fact that u_1^i is bounded in $W^{1,\infty}((0,T)\times\mathbb{R})$, for $i = 1, \cdot, \cdot, M$, and the inequality $|ab| \leq \frac{1}{2}(a^2 + b^2)$, we get :

$$
\begin{aligned}
|I_1| &\leq \frac{1}{2} M_1 (M+1) \|u_1\|_{[W^\infty((0,T)\times\mathbb{R})]^M} \int_{\mathbb{R}} \sum_{i=1,\dots,M} |u_1^i - u_2^i|^2 \psi^2, \\
&\leq \frac{1}{2} M_1 (M+1) \|u_1\|_{[W^\infty((0,T)\times\mathbb{R})]^M} \left(\sum_{i=1,\dots,M} \left\| (u_1^i - u_2^i)\psi \right\|^2_{L^2(\mathbb{R})} \right), \\
&\leq C \left(\sum_{i=1,\dots,M} \left\| (u_1^i - u_2^i)\psi \right\|^2_{L^2(\mathbb{R})} \right).
\end{aligned} \tag{9.61}
$$

Finally, (9.61), (9.59) and (9.60), imply :

$$
\frac{d}{dt} \left(\sum_{i=1,\dots,M} \left\| (u_1^i - u_2^i)\psi \right\|^2_{L^2(\mathbb{R})} \right) \leq 2 \left(|I_1| + |I_{21}| + |I_{22}| \right) \leq C \left(\sum_{i=1,\dots,M} \left\| (u_1^i - u_2^i)\psi \right\|^2_{L^2(\mathbb{R})} \right).
$$

Now, we apply the Gronwall Lemma and we use that $u_1^i(0,\cdot) = u_2^i(0,\cdot)$, to deduce that :

$$
\sum_{i=1,\dots,M} \left\| (u_1^i - u_2^i)\psi \right\|^2_{L^\infty((0,T);L^2(\mathbb{R}))} \leq \sum_{i=1,\dots,M} \left\| \left(u_1^i(0,\cdot) - u_2^i(0,\cdot) \right) \psi \right\|^2_{L^2(\mathbb{R})} e^{CT} = 0,
$$

i.e., $u_1 = u_2$ a.e in $(0,T)\times\mathbb{R}$. □

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Deuxième partie

Modèle bidimensionnel

Chapitre 5

Existence globale pour un système non-linéaire et non-local d'équations de transports décrivant la dynamique de densités de dislocations

Ce chapitre est un travail en collaboration avec M. Cannone, R. Monneau et F. Ribaud est issu de [25].

Dans ce travail, nous présentons un résultat d'existence globale pour le modèle de Groma-Balogh qui modélise la dynamique des densités de dislocations. Ce modèle est un problème bidimensionnel où les densités de dislocations satisfont un système d'équations de transport non-local. Plus précisément, le champ de vitesse dans ce système est la contrainte de cisaillement du matériau calculée à partir de l'équation de l'élasticité linéaire. Cette contrainte de cisaillement peut être exprimée commela transformation de Riesz des densités de dislocations. Le point clé de ce résultat est l'existence d'une estimation d'entropie sur le gradient des solutions.

Global existence for a system of non-linear and non-local transport equations describing the dynamics of dislocation densities

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Abstract

In this paper, we study the global in time existence problem for the Groma-Balogh model describing the dynamics of dislocation densities. This model is a bi-dimensional model where the dislocation densities satisfy a system of transport equations such that the velocity vector field is the shear stress in the material, solving the equations of elasticity. This shear stress can be expressed as some Riesz transform of the dislocation densities. The main tool in the proof of this result is the existence of an entropy for this system.

AMS Classification : 54C70, 35L45, 35Q72, 74H20, 74H25.
Key words : Cauchy's problem, system of non-linear transport equations, system of non-local transport equations, system of hyperbolic equations, entropy, Riesz transform, Zygmund space, dynamics of dislocation densities.

1 Introduction

1.1 Physical motivation and presentation of the model

Real crystals show certain defects in the organization of their crystalline structure, called dislocations. These defects were introduced in the Thirties by Taylor, Orowan and Polanyi as the principal explanation of plastic deformation at the microscopic scale of materials.

In a particular case where these defects are parallel lines in the three-dimensional space, their cross-section can be viewed as points in a plane. Under the effect of an exterior stress, dislocations can be moved. In the special case of what is called "edge dislocations", these dislocations move in the direction of their "Burgers vector" which has a fixed direction. (cf J. Hith and J. Lothe [74] for more physical description).

In this work, we are interested in the mathematical study of a model introduced by I. Groma, P. Balogh in [69] and [71]. In this model we consider two types of dislocations in the plane (x_1, x_2). Typically for a given velocity field, those dislocations of type $(+)$ propagate in the direction $+\vec{b}$ where $\vec{b} = (1, 0)$ is the Burgers vector, while those of type $(-)$ propagate in the direction $-\vec{b}$ (see Figure 1.1).

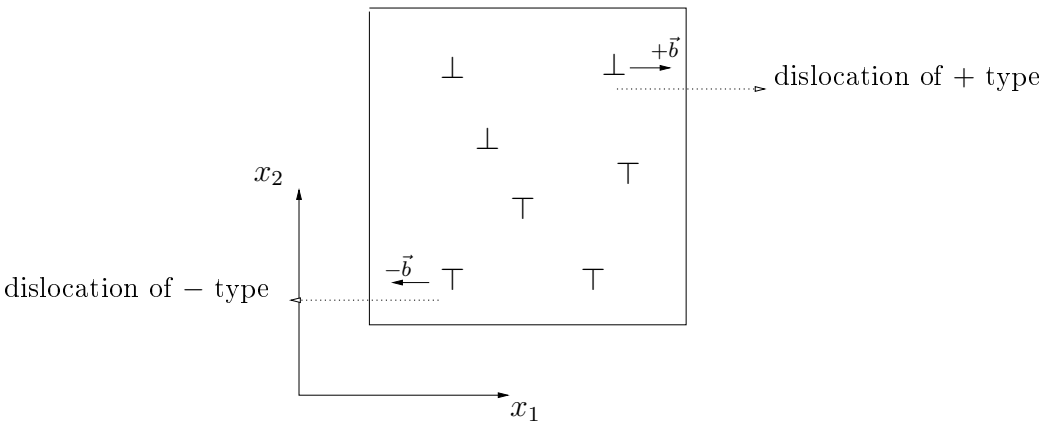

FIG. 5.1 – Groma-Balogh 2D model.

Here the velocity vector field is the shear stress in the material, solving the equations of elasticity. It turns out that this shear stress can be expressed as some Riesz transform of the solution (see Section 2). More precisely our non-linear and non-local system of transport equations is the following :

$$\begin{cases} \dfrac{\partial \rho^+}{\partial t}(x,t) = - \quad \left(R_1^2 R_2^2 \left(\rho^+(\cdot,t) - \rho^-(\cdot,t)\right)(x)\right) \dfrac{\partial \rho^+}{\partial x_1}(x,t) \quad \text{in } \mathcal{D}'(\mathbb{R}^2 \times (0,T)), \\ \dfrac{\partial \rho^-}{\partial t}(x,t) = \quad\quad \left(R_1^2 R_2^2 \left(\rho^+(\cdot,t) - \rho^-(\cdot,t)\right)(x)\right) \dfrac{\partial \rho^-}{\partial x_1}(x,t) \quad \text{in } \mathcal{D}'(\mathbb{R}^2 \times (0,T)). \end{cases} \tag{P}$$

The unknowns of the system (P) are the scalar functions ρ^+ and ρ^- at the time t and the position $x = (x_1, x_2)$, that we denote for simplification by $\rho^\pm$. These terms

correspond to the plastic deformations in a crystal. Their derivative in the x_1 direction (i.e. the direction of Burgers vector $\vec{b}$), $\dfrac{\partial \rho^{\pm}}{\partial x_1}$ represents the dislocation densities of $\pm$ type. The operators R_1 (resp. R_2) are the Riesz transformations associated to x_1 (resp. x_2) (for a precise definition of R_i, $i = 1, 2$, see Definition 1.1).

In fact, this 2D model has been generalized later in 2003 by I. Groma, F. Csikor and M. Zaiser in a model taking into account the back stress describing more carefully boundary layers (see [72] for further details). The Groma-Balogh model neglects in particular the short range dislocation-dislocation correlations in one slip direction. For an extension to multiple slip see S. Yefimov and E. Van der Giessen [135, ch. 5.] and [136]. This multiple slip version of the Groma-Balogh model presents some analogies with some traffic flow models (see O. Biham et al. [18] and J. Török, J. Kertész [130]). See also V. S. Deshpande et al. [42] for a similar model with boundary conditions and exterior forces. Recently, A. EL-Azab [46], M. Zaiser, T. Hochrainer [75], [137], [138] and R. Monneau [108] were interested in modeling the dynamics of dislocation densities in the three-dimensional space, but many more open questions have to be solved for establishing a satisfactory three-dimensional theory of dislocations dynamics and for getting rigorous results.

From a technical point of view, (P) is related to other well known models, such as the transport equation with a low regularity vector field. This equation was studied in the work of R. J. Diperna, P. L. Lions [45] and L. Ambrosio [8], where the authors showed the existence and uniqueness of renormalized solutions by considering vector fields in $L^1((0,T); W^{1,1}_{loc}(\mathbb{R}^N))$ and $L^1((0,T); BV_{loc}(\mathbb{R}^N))$ respectively in both cases with bounded divergence. On the contrary in system (P), we are only able to prove that for the constructed solution, the vector field is in $L^2((0,T); W^{1,2}_{loc}(\mathbb{R}^2))$ without any better estimate on the divergence of the vector field.

We stress out the attention of the reader that there was no any existence and uniqueness result for (P) . In this paper we prove that (P) admits a "global in time" solution.

More generally in the frame of symmetric hyperbolic system, we refer to the book of D. Serre [123, Vol I, Th 3.6.1], for a typical result of local existence and uniqueness in $C([0,T); H^s(\mathbb{R}^N)) \cap C^1([0,T); H^{s-1}(\mathbb{R}^N))$, with $s > \frac{N}{2} + 1$, by considering initial data in $H^s(\mathbb{R}^N)$. This result remains local in time, even in dimension $N = 2$.

We can also remark that in the case where we multiply the right side of the two equations in system (P) by -1, we get a quasi-geostrophic-like system. For those who are concerned in quasi-geostrophic systems, we refer to P. Constantin et al. [34], to [35] for certain 2D numerical results. We also refer to J. Wu [133, Th 4.1] for 2D

local existence and uniqueness results in Hölder spaces and to A. Córdoba, D. Córdoba [36], D. Chae, A. Córdoba [29] for blow-up results in finite time, in dimension one.

Let us also mention some related Vlasov-Poisson models (see J. Nieto et al. [111] for instance) and a related model in superconductivity studied by N. Masmoudi et al. [107] and by L. Ambrosio et al. [9]. These models were derived from some Vlasov-Poison-Fokker-Planck models (see for instance T. Goudon et al. [67], and P. Chavanis et al. [30] for an overview of similar models). It is also worth mentioning that this model is related to Vlasov-Navier-Stokes equation see T. Goudon et al. [65], [66].

1.2 Main result

In the present paper, we prove a "global in time" existence result for the system (P) describing the dynamics of dislocation densities.
In this work we consider the following initial conditions :

$$\rho^{\pm}(x_1, x_2, t = 0) = \rho_0^{\pm}(x_1, x_2) = \rho_0^{\pm,per}(x_1, x_2) + Lx_1, \tag{IC}$$

where $\rho^{\pm,per}$ is a 1-periodic function in x_1 and x_2. The periodicity is a way of studying the bulk behavior of the material away from its boundary. Here L is a given positive constant that represents the initial total dislocation densities of $\pm$ type on the periodic cell.

First of all we give some results which prove that the bilinear term in (P) is well defined.

Definition 1.1 (Riesz transform)
Let $p > 1$ and $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, the periodic square $[0,1) \times [0,1)$. If $f \in L^p(\mathbb{T}^2)$, we define R_i for $i \in \{1,2\}$ as the Riesz transforms over $\mathbb{T}^2$ such that the Fourier series coefficients are given by :
i) $c_{(0,0)}(R_i f) = 0$,
ii) $c_k(R_i f) = \dfrac{k_i}{|k|} c_k(f)$ *for* $k = (k_1, k_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$,
where $c_k(f) = \displaystyle\int_{\mathbb{T}^2} f(x) e^{-2\pi i k \cdot x} d^2x$.

Definition 1.2 (The space $L \log L$)
We define $L \log L(\mathbb{T}^2)$ as the following special case of Zygmund spaces (see C. Bennett and R. Sharpley [16, Page 243]) :

$$L \log L(\mathbb{T}^2) = \left\{ f \in L^1(\mathbb{T}^2) \text{ such that } \int_{\mathbb{T}^2} |f| \log(e + |f|) < +\infty \right\}.$$

where log *denotes the neperian logarithm. This space is endowed with the norm*

$$\|f\|_{L\log L(\mathbb{T}^2)} = \inf\left\{\lambda > 0 : \int_{\mathbb{T}^2} \frac{|f|}{\lambda} \log\left(e + \frac{|f|}{\lambda}\right) \leq 1\right\},$$

which is due to Luxemburg (see R. A. Adams [2, (13), Page 234]).

For other equivalent definitions of Zygmund spaces (see P. Koosis [90, Page 96], E. M. Stein [126, Page 43] and A. Zygmund [139]). We now present the following proposition.

Proposition 1.3 (Meaning of the bilinear term)
Let $T > 0$, f and g be two functions defined on $\mathbb{T}^2 \times (0,T)$, such that $f \in L^1((0,T); W^{1,2}(\mathbb{T}^2))$ and $g \in L^\infty((0,T); L\log L(\mathbb{T}^2))$ then,

$$fg \in L^1(\mathbb{T}^2 \times (0,T)).$$

The proof of this proposition is given in Subsection 4.2. We can now state our main result.

Theorem 1.4 (Global existence)
For all $T, L > 0$, and for every initial data $\rho_0^\pm \in L^2_{loc}(\mathbb{R}^2)$ with
(H1) $\rho_0^\pm(x_1+1, x_2) = \rho_0^\pm(x_1, x_2) + L$, *a.e. in* $\mathbb{R}^2$,
(H2) $\rho_0^\pm(x_1, x_2+1) = \rho_0^\pm(x_1, x_2)$, *a.e. in* $\mathbb{R}^2$,
(H3) $\dfrac{\partial {\rho_0}^\pm}{\partial x_1} \geq 0$, *a.e. in* $\mathbb{R}^2$,
(H4) $\left\|\dfrac{\partial {\rho_0}^\pm}{\partial x_1}\right\|_{L\log L(\mathbb{T}^2)} \leq C$, *with* $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$,
the system (P)-(IC) admits solutions $\rho^\pm \in L^\infty((0,T); L^2_{loc}(\mathbb{R}^2)) \cap C([0,T); L^1_{loc}(\mathbb{R}^2))$ in the distributional sense. These solutions satisfy $(H1)$, $(H2)$, $(H3)$ and $(H4)$ for a.e. $t \in (0,T)$. Moreover, we have :
(P1) $R_1^2 R_2^2 (\rho^+ - \rho^-) \in L^2((0,T); W^{1,2}_{loc}(\mathbb{R}^2))$.

Remark 1.5 (Bilinear term)
It is clear here that the bilinear term is always defined via $(P1)$ and Proposition 1.3.

Remark 1.6 (Entropy and energy inequalities)
It turns out that the constructed solution also satisfies the following fundamental entropy inequality (as a consequence of Lemma 7.7), for a.e. $t \in (0,T)$,

$$\int_{\mathbb{T}^2} \sum_{\pm} \frac{\partial \rho^\pm}{\partial x_1}(\cdot,t) \log\left(\frac{\partial \rho^\pm}{\partial x_1}(\cdot,t)\right) + \int_0^t \int_{\mathbb{T}^2} \left(R_1 R_2 \left(\frac{\partial \rho^+}{\partial x_1} - \frac{\partial \rho^-}{\partial x_1}\right)\right)^2 \leq C_1, \quad (1.1)$$

with $C_1 = C_1\left(\left\|\frac{\partial \rho_0{}^{\pm}}{\partial x_1}\right\|_{L\log L(\mathbb{T}^2)}\right)$.

Moreover, (at least formally for enough regular functions) the following energy inequality holds :

$$\frac{1}{2}\int_{\mathbb{T}^2}\left(R_1R_2(\rho^+-\rho^-)(\cdot,t)\right)^2+\int_0^t\int_{\mathbb{T}^2}\left(R_1^2R_2^2(\rho^+-\rho^-)\right)^2\left(\frac{\partial\rho^+}{\partial x_1}+\frac{\partial\rho^-}{\partial x_1}\right)\leq C_2,$$

with $C_2 = C_2\left(\left\|\rho_0^+-\rho_0^-\right\|_{L^2(\mathbb{T}^2)}\right)$.

Remark 1.7 (Bounds on the solution)
If we denote $\rho=\rho^+-\rho-$, *then there exists a constant* C *independent on* T, *and a constant* C_T *depending on* T *such that,*

$(E1)$ $\|\rho^{\pm}-Lx_1\|_{L^\infty((0,T);L^2(\mathbb{T}^2))}\leq C_T$, $\quad$ $(E2)$ $\|R_1^2R_2^2\rho\|_{L^\infty((0,T);BMO(\mathbb{T}^2))}\leq C$,

$(E3)$ $\left\|\frac{\partial\rho^{\pm}}{\partial x_1}\right\|_{L^\infty((0,T);L\log L(\mathbb{T}^2))}\leq C$, $\quad$ $(E4)$ $\|R_1^2R_2^2\rho\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))}\leq C$,

$(E5)$ $\left\|\frac{\partial\rho^{\pm}}{\partial t}\right\|_{L^2((0,T);W^{-2,2}(\mathbb{T}^2))}\leq C_T$, $\quad$ $(E6)$ $\left\|R_1^2R_2^2\frac{\partial\rho}{\partial t}\right\|_{L^2((0,T);W^{-1,2}(\mathbb{T}^2))}\leq C_T$,

where $W^{-1,2}(\mathbb{T}^2)$ *and* $W^{-2,2}(\mathbb{T}^2)$ *are respectively the dual spaces of* $W^{1,2}(\mathbb{T}^2)$ *and* $W^{2,2}(\mathbb{T}^2)$. *The space* BMO *is the set of bounded mean oscillation functions that will be precised in the sequel (see Definition 7.1).*

In order to prove our main theorem we regularize the system (P) by the mean of the viscosity term ($\varepsilon\Delta\rho^{\pm}$) and the initial data (IC) by classical convolution. Then, using a fixed point Theorem, we prove that our regularized system admits local in time solutions. Moreover, as we get some ε-independent *a priori* estimates we will be able to extend our local in time solution into a global one. This turns out to be possible thanks to the entropy inequality (1.1). Then, joined with other *a priori* estimates, it will be possible to prove some compactness properties and pass to the limit as ε goes to 0 is the ε-problem.

In a particular sub-case of this model where the dislocation densities depend on a single variable $x = x_1 + x_2$, the existence and uniqueness of a Lipschitz viscosity solution was proved in A. El Hajj, N. Forcadel [48]. Also the existence and uniqueness of a strong solution in $W^{1,2}_{loc}(\mathbb{R}\times[0,T))$ was proved in A. El Hajj [47]. Concerning the model of I. Groma, F. Csikor, M. Zaiser [72] which takes into consideration the short range dislocation-dislocation correlations giving a parabolic-hyperbolic system, let

us mention the work of H. Ibrahim [78] where a result of existence and uniqueness of a viscosity solution is given but only for a one-dimensional model.

Our study of the dynamics of dislocation densities in a special geometry is related to the more general dynamics of dislocation lines. We refer the interested reader to the work of O. Alvarez et al. [7], for a local existence and uniqueness of some non-local Hamilton-Jacobi equation. We also refer to O. Alvarez et al. [3] and G. Barles, O. Ley [15] for some long time existence results.

1.3 Organization of the paper

First, in Section 2, we recall the physical derivation of system (P). In Section 3, we give our notation for the sequel of the paper. In Section 4, we give the proof of Proposition 1.3. We also prove that the bilinear term of our system has a better mathematical meaning (see Proposition 4.6). Next, in Section 5, we regularize the initial conditions and we prove that the system (P), modified by a term $(\varepsilon\Delta\rho^{\pm})$, admits local in time solutions (in the "Mild" sense). This will be achieved by using an application of a fixed point Theorem. In Section 6, we prove that the obtained solutions are regular and increasing for all $t \in (0, T)$, for increasing initial data. In Section 7, we prove some ε-uniform *a priori* estimates for the regularized solution obtained in Section 6. Then thanks to these *a priori* estimates, we prove in Section 8 that the local in time solutions constructed in Section 6 are in fact global in time for the ε-problem. Finally, in Section 9, we achieve the proof of our main Theorem, passing to the limit in the equation as ε goes to 0, and using some compactness properties inherited from our a priori estimates.

2 Physical derivation of the model

In this section we explain how to get physically the system (P). We consider a three-dimensional crystal, with displacement

$$u = (u_1, u_2, u_3) : \mathbb{R}^3 \to \mathbb{R}^3.$$

For $x = (x_1, x_2, x_3)$, and an orthogonal basis (e_1, e_2, e_3), we define the total strain by :

$$\varepsilon(u) = \frac{1}{2}(\nabla u + {}^t\nabla u), \quad \text{i.e.} \quad \varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2, 3.$$

This total strain is decomposed as

$$\varepsilon(u) = \varepsilon^e(u) + \varepsilon^p,$$

with $\varepsilon^e(u)$ is the elastic strain and ε^p the plastic strain which is defined by :

$$\varepsilon^p = \varepsilon^0 \gamma, \tag{2.2}$$

with $\varepsilon^0 = \dfrac{1}{2}\left(\vec{e_1} \otimes \vec{e_2} + \vec{e_2} \otimes \vec{e_1}\right)$ in the special case of a single slip system where dislocations move in the plane $\{x_3 = 0\}$ with Burgers vector $\vec{b} = e_1$. Here γ is the resolved plastic strain, and will be precised later. The stress in the crystal satisfies the equation of elasticity div $\sigma = 0$ and is given by,

$$\sigma = \Lambda : \varepsilon^e(u), \tag{2.3}$$

where for $i, j = 1, 2, 3$,

$$(\Lambda : \varepsilon^e(u))_{ij} = 2\mu\varepsilon^e_{ij}(u) + \lambda\delta_{ij} tr(\varepsilon^e(u)), \tag{2.4}$$

and $\lambda, \mu > 0$ are the constants of Lamé coefficients of the crystal (here, for simplification, assumed isotropic).
We now assume that we are in a particular geometry where the dislocations are straight lines parallel to the direction e_3 and that the problem is invariant by translation in the x_3 direction. Moreover we assume that $u_3 = 0$. Then, this problem reduces to a bi-dimensional problem with u_1, u_2 only depending on (x_1, x_2) and so we can express the resolved plastic strain γ as

$$\gamma = \rho^+ - \rho^-,$$

where $\dfrac{\partial \rho^+}{\partial x_1}$ and $\dfrac{\partial \rho^-}{\partial x_1}$ are respectively the densities of dislocations of Burgers vectors given by $\vec{b} = e_1$ and $\vec{b} = -e_1$.
Furthermore, these dislocation densities are transported in the direction of the Burgers vectors by a velocity. This velocity is given by the resolved shear stress $(\sigma : \varepsilon^0)$ up to sign of the Burgers vectors. More precisely, we have :

$$\frac{\partial \rho^\pm}{\partial t} = \pm(\sigma : \varepsilon^0) e_1 . \nabla \rho^\pm.$$

Finally, the functions $\rho^\pm$ and u are solutions of the coupled system (see I. Groma, P. Balogh [71], [69]),

$$\begin{cases} \text{div } \sigma & = 0 & \text{in } \mathbb{R}^2 \times (0,T), \\ \sigma & = \Lambda : (\varepsilon(u) - \varepsilon^p) & \text{in } \mathbb{R}^2 \times (0,T), \\ \varepsilon(u) & = \frac{1}{2}\left(\nabla u + {}^t\nabla u\right) & \text{in } \mathbb{R}^2 \times (0,T), \\ \varepsilon^p & = \varepsilon^0\left(\rho^+ - \rho^-\right) & \text{in } \mathbb{R}^2 \times (0,T), \\ \dfrac{\partial \rho^\pm}{\partial t} & = \pm(\sigma : \varepsilon^0) e_1 . \nabla \rho^\pm & \text{in } \mathbb{R}^2 \times (0,T), \end{cases} \tag{2.5}$$

i.e in coordinates,

$$\begin{cases} \displaystyle\sum_{j=1,2} \frac{\partial \sigma_{ij}}{\partial x_j} & = 0 & \text{in } \mathbb{R}^2 \times (0,T), \\ \sigma_{ij} & = 2\mu\varepsilon^e_{ij}(u) + \lambda\delta_{ij} tr(\varepsilon^e(u)) & \text{in } \mathbb{R}^2 \times (0,T), \\ \varepsilon_{ij}(u) & = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right) & \text{in } \mathbb{R}^2 \times (0,T), \\ \varepsilon^p_{ij} & = \dfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\left(\rho^+ - \rho^-\right) & \text{in } \mathbb{R}^2 \times (0,T), \\ \dfrac{\partial \rho^\pm}{\partial t} & = \pm\sigma_{12} e_1 . \nabla \rho^\pm & \text{in } \mathbb{R}^2 \times (0,T), \end{cases} \tag{2.6}$$

where the unknowns of the system are $\rho^\pm$ and the displacement (u_1, u_2).

Then the following lemma holds.

***Lemma* 2.1 (Equivalence between 2-D systems)**
Assume that (u_1, u_2) and $\rho^+ - \rho^-$ are $\mathbb{Z}^2$-periodic functions. Then the 2-D problem (2.6), is equivalent to the following 2-D problem

$$\begin{cases} \dfrac{\partial \rho^+}{\partial t} = - \quad C_1\left(R_1^2 R_2^2\left(\rho^+ - \rho^-\right)\right)\dfrac{\partial \rho^+}{\partial x_1} & \textit{in } \mathbb{R}^2 \times (0,T), \\ \dfrac{\partial \rho^-}{\partial t} = \quad C_1\left(R_1^2 R_2^2\left(\rho^+ - \rho^-\right)\right)\dfrac{\partial \rho^-}{\partial x_1} & \textit{in } \mathbb{R}^2 \times (0,T), \end{cases} \tag{2.7}$$

where $C_1 = 4\dfrac{(\lambda+\mu)\mu}{\lambda+2\mu}$.

As the constant C_1 is non-negative, rescaling in time in system (2.7), we can replace this constant by 1.

Proof of Lemma 2.1 :
We can rewrite the first equation of (2.5) as

$$\operatorname{div}\left(2\mu\varepsilon(u) + \lambda tr(\varepsilon(u))I_d\right) = \operatorname{div}\left(2\mu\varepsilon^p + \lambda tr(\varepsilon^p)I_d\right).$$

This implies that :

$$\mu\Delta u + (\lambda+\mu)\nabla(\operatorname{div} u) = \mu\begin{pmatrix} \dfrac{\partial}{\partial x_2}(\rho^+ - \rho^-) \\ \\ \dfrac{\partial}{\partial x_1}(\rho^+ - \rho^-) \end{pmatrix}. \tag{2.8}$$

We now derive the first equation and the second equation of the previous system with respect to x_1 and x_2 respectively. We obtain

$$\mu\Delta\begin{pmatrix}\dfrac{\partial u_1}{\partial x_1}\\[2ex] \dfrac{\partial u_2}{\partial x_2}\end{pmatrix}+(\lambda+\mu)\begin{pmatrix}\dfrac{\partial^2}{\partial x_1^2}(\text{div } u)\\[2ex] \dfrac{\partial^2}{\partial x_2^2}(\text{div } u)\end{pmatrix}=\mu\begin{pmatrix}\dfrac{\partial^2}{\partial x_1\partial x_2}(\rho^+-\rho^-)\\[2ex] \dfrac{\partial^2}{\partial x_2\partial x_1}(\rho^+-\rho^-)\end{pmatrix}.$$

Now, by adding the two above equations, we get

$$(\lambda+2\mu)\Delta(\text{div } u)=2\mu\frac{\partial^2}{\partial x_1\partial x_2}(\rho^+-\rho^-).$$

Applying Δ^{-1} to this expression we get (div u) that we plug it into (2.8). Which leads to

$$\Delta u=\begin{pmatrix}\dfrac{\partial}{\partial x_2}(\rho^+-\rho^-)\\[2ex] \dfrac{\partial}{\partial x_1}(\rho^+-\rho^-)\end{pmatrix}-2\frac{(\lambda+\mu)}{(\lambda+2\mu)}\nabla\Delta^{-1}\frac{\partial^2}{\partial x_1\partial x_2}(\rho^+-\rho^-). \tag{2.9}$$

As, previously, we derive the first equation and the second equation of system (2.10) with respect to x_2 and x_1 respectively, and obtain

$$\Delta\begin{pmatrix}\dfrac{\partial u_1}{\partial x_2}\\[2ex] \dfrac{\partial u_2}{\partial x_1}\end{pmatrix}=\begin{pmatrix}\dfrac{\partial^2}{\partial x_2^2}(\rho^+-\rho^-)\\[2ex] \dfrac{\partial^2}{\partial x_1^2}(\rho^+-\rho^-)\end{pmatrix}-2\frac{(\lambda+\mu)}{(\lambda+2\mu)}\Delta^{-1}\begin{pmatrix}\dfrac{\partial^4}{\partial x_1^2\partial x_2^2}(\rho^+-\rho^-)\\[2ex] \dfrac{\partial^4}{\partial x_1^2\partial x_2^2}(\rho^+-\rho^-)\end{pmatrix}. \tag{2.10}$$

Now, adding the two above equations, we infer that

$$\Delta\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)=\Delta(\rho^+-\rho^-)-4\frac{(\lambda+\mu)}{(\lambda+2\mu)}\Delta^{-1}\frac{\partial^4}{\partial x_1^2\partial x_2^2}(\rho^+-\rho^-). \tag{2.11}$$

Using that

$$(\sigma:\varepsilon^0)=\sigma_{12}=2\mu(\varepsilon^e(u))_{12}=\mu\left(\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)-(\rho^+-\rho^-)\right), \tag{2.12}$$

together with equation (2.11), this yields

$$(\sigma : \varepsilon^0) = -4\frac{(\lambda+\mu)\mu}{(\lambda+2\mu)}\Delta^{-2}\frac{\partial^4}{\partial x_1^2 \partial x_2^2}(\rho^+ - \rho^-) = -C_1\left(R_1^2 R_2^2(\rho^+ - \rho^-)\right).$$

Hence we see that the system (2.5) can be rewritten as (2.7). □

Remark 2.2 (Property of the elastic energy)
If we define the elastic energy by

$$E = \frac{1}{2}\int_{\mathbb{R}^2/\mathbb{Z}^2} \Lambda : (\varepsilon^e(u) : \varepsilon^e(u)),$$

with $\varepsilon^e(u) = \varepsilon(u) - \varepsilon^0(\rho^+ - \rho^-)$. *Then, since by the equation of elasticity* $\dfrac{\partial E}{\partial u} = 0$, *we can notice if* $\dfrac{\partial \rho^+}{\partial x_1}, \dfrac{\partial \rho^-}{\partial x_1} \geq 0$ *that,*

$$\frac{dE}{dt} = -\int_{\mathbb{R}^2/\mathbb{Z}^2} (\Lambda : \varepsilon^e(u)) : \varepsilon^0 \frac{\partial(\rho^+ - \rho^-)}{\partial t} = -\int_{\mathbb{R}^2/\mathbb{Z}^2} \sigma_{12}^2 \left(\frac{\partial \rho^+}{\partial x_1} + \frac{\partial \rho^-}{\partial x_1}\right) \leq 0.$$

This formal result indicates that the elastic energy is a non-increasing quantity in this model. Hence, the elastic energy E *is a Lyapunov functional for our dissipative model.*

3 Notation

In what follows, we are going to use the following notation :

1. $\rho = \rho^+ - \rho^-$,
2. $\rho^{\pm,per}(x_1, x_2, t) = \rho^{\pm}(x_1, x_2, t) - Lx_1$,
3. $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the periodic interval $[0, 1)$, and $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ the periodic square $[0, 1) \times [0, 1)$.
4. Let f be a function defined on $\mathbb{R}^2 \times (0, T)$ having values in $\mathbb{R}^2$, we denote by $f(t) = f(., t) : x \longmapsto f(x, t)$.
5. We write $\int_{\mathbb{T}}$ in place of $\int_0^1$
6. Let E be a Banach space and $f = (f_1, f_2)$ a vector such that $f_i \in E$ for $i \in \{1, 2\}$. The norm of f in E^2 will be defined as $\|f\|_{E^2} = \max(\|f_1\|_E, \|f_2\|_E)$.
7. Throughout the paper, C is an arbitrary positive constant.

4 Concerning the meaning of the solution of (P)

In this Section we prove Proposition 1.3. This shows that if (P) admits solutions verifying the conditions of Theorem 1.4, then we can give a mathematical meaning to the bilinear term. In order to do this, we need to define some functional spaces and recall some of their properties, that will be used later in our work.

4.1 Properties of some useful Orlicz spaces

We recall the definition of Orlicz spaces and some of their properties. For details, we refer to R. A. Adams [2, Ch. 8] and M. M. Rao, Z. D. Ren [120].
A real valued function $A : [0, +\infty) \to \mathbb{R}$ is called a Young function if it has the following properties (see R. O'Neil [114, Def 1.1]) :

- A is a continuous, non-negative, non-decreasing and convex function.
- $A(0) = 0$ and $\lim_{t\to+\infty} A(t) = +\infty$.

Let $A(\cdot)$ be a Young function. The Orlicz class $K_A(\mathbb{T}^2)$ is the set of (equivalence classes of) real-valued measurable function h on $\mathbb{T}^2$ satisfying

$$\int_{\mathbb{T}^2} A(|h(x)|) < +\infty.$$

The Orlicz space $L_A(\mathbb{T}^2)$ is the linear hull of the Orlicz class $K_A(\mathbb{T}^2)$, that is, the smallest vector space (under pointwise addition and scalar multiplication) containing $K_A(\mathbb{T}^2)$. Evidently $L_A(\mathbb{T}^2)$ consists of all scalar multiples λh of elements $h \in K_A(\mathbb{T}^2)$. These Orlicz space supplemented with the Luxemburg norm,

$$\|f\|_{L_A(\mathbb{T}^2)} = \inf\left\{\lambda > 0 : \int_{\mathbb{T}^2} A\left(\frac{|h(x)|}{\lambda}\right) \leq 1\right\}.$$

Endowed with this norm, the Orlicz space $L_A(\mathbb{T}^2)$ is a Banach space. For example if $A(t) = t^p$ for $p \geq 1$, the Orlicz space is the usual Lebesgue space $L^p(\mathbb{T}^2)$.

Remark 4.1 (Separability)
If A is Δ_2-regular (i.e. there exists a positive constant δ such that for all $t \geq 0$, $A(2t) \leq \delta A(t)$), then the Orlicz space $L_A(\mathbb{T}^2)$ is separable (see M. M. Rao and Z. D. Ren [120, Th 1, Page 87]). In particular this holds for $L\log L(\mathbb{T}^2)$.

Definition 4.2 (Some Orlicz spaces)
For $\alpha \geq 1$, we denote by

$$EXP_\alpha(\mathbb{T}^2), \text{ the Orlicz space defined by the function } A(t) = e^{t^\alpha} - 1.$$

Another space of interest will be the Zygmund space

$L\log^\beta L(\mathbb{T}^2)$, *the Orlicz space defined by the function* $A(t) = t(\log(e+t))^\beta$, *for* $\beta \geq 0$.

Observe that those spaces are Banach spaces and that $EXP_{\frac{1}{\beta}}(\mathbb{T}^2)$ is the dual of $L\log^\beta L(\mathbb{T}^2)$, for $0 < \beta \leq 1$ (see C. Bennett and R. Sharpley [16, Def 6.11]). It is worth noticing that $L\log^1 L(\mathbb{T}^2) = L\log L(\mathbb{T}^2)$.

Let us recall some useful properties of these spaces. The first one is the generalized Hölder inequality.

***Lemma* 4.3 (Generalized Hölder inequality)**
i) Let $f \in EXP_2(\mathbb{T}^2)$ *and* $g \in L\log^{\frac{1}{2}} L(\mathbb{T}^2)$, *Then there exists a constant* C *such that (see R. O'Neil [114, Th 2.3]),*

$$\|fg\|_{L^1(\mathbb{T}^2)} \leq C\|f\|_{EXP_2(\mathbb{T}^2)}\|g\|_{L\log^{\frac{1}{2}} L(\mathbb{T}^2)}.$$

ii) Let $f \in EXP_2(\mathbb{T}^2)$ *and* $g \in L\log L(\mathbb{T}^2)$. *Then there exists a constant* C *such that (see R. O'Neil [114, Th 2.3]),*

$$\|fg\|_{L\log^{\frac{1}{2}} L(\mathbb{T}^2)} \leq C\|f\|_{EXP_2(\mathbb{T}^2)}\|g\|_{L\log L(\mathbb{T}^2)}.$$

The proof of this lemma is given in Appendix.
The second property is the Trudinger embedding,

***Lemma* 4.4 (Continuous Trudinger embedding)**
We have the following continuous injection (see N. S. Trudinger [129] and R. A. Adams [2, Th 8. 25]) :

$$W^{1,2}(\mathbb{T}^2) \hookrightarrow EXP_2(\mathbb{T}^2).$$

Finally, we have the following embedding.

***Lemma* 4.5 (Properties of the Zygmund space)**
For $1 < p < +\infty$, $\alpha \geq 1$ *and* $\beta \geq 0$ *we have the following continuous embedding :*

$$L^\infty(\mathbb{T}^2) \hookrightarrow EXP_\alpha(\mathbb{T}^2) \hookrightarrow L^p(\mathbb{T}^2) \hookrightarrow L\log^\beta L(\mathbb{T}^2) \hookrightarrow L^1(\mathbb{T}^2).$$

For the proof, see for instance R. A. Adams [2, Th 8.12].

4.2 Sharp estimate of the bilinear term

Now, we propose to verify with the help of the following proposition that the system (P) has indeed a sense, and first prove a better estimate than those mentioned in Proposition 1.3. Namely, we have the following.

Proposition 4.6 (Estimate of the bilinear term)
Let $T > 0$, f and g be two functions defined on $\mathbb{T}^2 \times (0,T)$, such that,

(1) $f \in L^2((0,T); W^{1,2}(\mathbb{T}^2))$,

(2) $g \in L^\infty((0,T); L\log L(\mathbb{T}^2))$. *Then,*

$$fg \in L^2((0,T); L\log^{\frac{1}{2}} L(\mathbb{T}^2)),$$

and for a positive constant C, we have :

$$\|fg\|_{L^2((0,T);L\log^{\frac{1}{2}} L(\mathbb{T}^2))} \leq C\|f\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))}\|g\|_{L^\infty((0,T);L\log L(\mathbb{T}^2))}.$$

Proof of Proposition 4.6 :
First of all, according to the generalized Hölder inequality Lemma 4.3 (ii), we know that

$$\|f(t)g(t)\|^2_{L\log^{\frac{1}{2}} L(\mathbb{T}^2)} \leq C\|f(t)\|^2_{EXP_2(\mathbb{T}^2)}\|g(t)\|^2_{L\log L(\mathbb{T}^2)}.$$

Integrating on $(0,T)$, we infer that,

$$\int_0^T \|f(t)g(t)\|^2_{L\log^{\frac{1}{2}} L(\mathbb{T}^2)} \leq C\int_0^T \|f(t)\|^2_{EXP_2(\mathbb{T}^2)}\|g(t)\|^2_{L\log L(\mathbb{T}^2)}.$$

Knowing that $g \in L^\infty((0,T); L\log L(\mathbb{T}^2))$, we have,

$$\|fg\|^2_{L^2((0,T);L\log^{\frac{1}{2}} L(\mathbb{T}^2))} \leq C\|g\|^2_{L^\infty((0,T);L\log L(\mathbb{T}^2))}\|f\|^2_{L^2((0,T);EXP_2(\mathbb{T}^2))}.$$

Now, by the Trudinger inequality Lemma 4.4, we get,

$$\|fg\|^2_{L^2((0,T);L\log^{\frac{1}{2}} L(\mathbb{T}^2))} \leq C\|g\|^2_{L^\infty((0,T);L\log L(\mathbb{T}^2))}\|f\|^2_{L^2((0,T);W^{1,2}(\mathbb{T}^2))}.$$

□

Proof of Proposition 1.3 :

We proceed as in the proof of Proposition 4.6. We use Lemma 4.3 (i), and integrate in time, thanks to the Trudinger inequality (Lemma 4.4) and the continuous injection $L\log L(\mathbb{T}^2) \hookrightarrow L\log^{\frac{1}{2}} L(\mathbb{T}^2)$. □

5 Local existence of solutions of a regularized system

In this Section, we prove a local in time existence for the system (P), modified by the term $\varepsilon\Delta\rho^{\pm}$, and for smoothed data. This modification brings us to study, for all $0 < \varepsilon \leq 1$, the following system :

$$\begin{cases} \dfrac{\partial \rho^{+,\varepsilon}}{\partial t} - \varepsilon\Delta\rho^{+,\varepsilon} &= -(R_1^2 R_2^2 \rho^{\varepsilon})\dfrac{\partial \rho^{+,\varepsilon}}{\partial x_1} \quad \text{in } \mathcal{D}'(\mathbb{R}^2 \times (0,T)), \\ \dfrac{\partial \rho^{-,\varepsilon}}{\partial t} - \varepsilon\Delta\rho^{-,\varepsilon} &= \quad (R_1^2 R_2^2 \rho^{\varepsilon})\dfrac{\partial \rho^{-,\varepsilon}}{\partial x_1} \quad \text{in } \mathcal{D}'(\mathbb{R}^2 \times (0,T)), \end{cases} \tag{P_ε}$$

where $\rho^{\varepsilon} = \rho^{+,\varepsilon} - \rho^{-,\varepsilon}$, with the following regular initial data :

$$\rho^{\pm,\varepsilon}(x,0) = \rho_0^{\pm,\varepsilon}(x) = \rho_0^{\pm,per} * \eta_\varepsilon(x) + (L+\varepsilon)x_1 = \rho_0^{\pm,\varepsilon,per}(x) + L_\varepsilon x_1, \tag{IC_ε}$$

where $\eta_\varepsilon(\cdot) = \frac{1}{\varepsilon^2}\eta(\frac{\cdot}{\varepsilon})$, such that $\eta \in C_c^\infty(\mathbb{R}^2)$ is a non-negative function and $\int_{\mathbb{R}^2}\eta = 1$.

Remark 5.1
We consider L_ε to obtain strongly monotonous initial data $\rho_0^{\pm,\varepsilon}$. This condition will be useful in the proof of Lemma 7.7.

If we let $\rho^{\pm,\varepsilon,per} = \rho^{\pm,\varepsilon} - L_\varepsilon x_1$, we know that the system (P_ε) is equivalent to,

$$\frac{\partial \rho^{\pm,\varepsilon}}{\partial t} - \varepsilon\Delta\rho^{\pm,\varepsilon,per} = \mp(R_1^2 R_2^2 \rho^{\varepsilon})\frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x_1} \mp L_\varepsilon(R_1^2 R_2^2 \rho^{\varepsilon}) \text{ in } \mathcal{D}'(\mathbb{T}^2 \times (0,T)), \tag{P_ε^{per}}$$

with initial conditions,

$$\rho^{\pm,\varepsilon,per}(x,0) = \rho_0^{\pm,\varepsilon}(x) - L_\varepsilon x_1 = \rho_0^{\pm,\varepsilon,per}(x). \tag{IC_ε^{per}}$$

Remark 5.2
The properties of the mollifier $(\eta_\varepsilon)_\varepsilon$ and the fact that $\rho_0^{\pm,per} \in L^2(\mathbb{T}^2)$ implies that $\rho_0^{\pm,\varepsilon,per} \in C^\infty(\mathbb{T}^2)$. In particular, $\rho_0^{\pm,\varepsilon,per} \in W^{1,p}(\mathbb{T}^2)$ for all $1 \leq p \leq +\infty$.

The following theorem is a local existence result (in the "Mild" sense) of the regularized system (P_ε)-(IC_ε). This result is achieved in a super-critical space. Here particularly we chose the space of functions $C([0,T); W_{loc}^{1,\frac{3}{2}}(\mathbb{R}^2))$. Later, in Section 6, we will improve the regularity of the solution.

Theorem 5.3 (Local existence result)
For all initial data $\rho_0^{\pm} \in L^2_{loc}(\mathbb{R}^2)$ verifying $(H1)$ and $(H2)$, there exists

$$T^\star(\|\rho_0^{\pm,\varepsilon,per}\|_{W^{1,\frac{3}{2}}(\mathbb{T}^2)}, L, \varepsilon) > 0,$$

such that the system (P_ε)-(IC_ε) admits solutions $\rho^{\pm,\varepsilon} \in C([0,T^\star); W^{1,\frac{3}{2}}_{loc}(\mathbb{R}^2))$, satisfying $(H1)$ and $(H2)$ for a.e. $t \in (0,T^\star)$.

Before proving Theorem 5.3, let us recall some well known results.

5.1 Useful results

We first start with reformulation of system (P_ε^{per})-(IC_ε^{per}) as an integral system.

***Lemma* 5.4 (Mild solutions are solutions in the distributional sense)**
If $\rho^{\pm,\varepsilon,per} \in C([0,T); W^{1,\frac{3}{2}}(\mathbb{T}^2))$ are solutions of the following integral problem :

$$\begin{aligned} \rho^{\pm,\varepsilon,per}(\cdot,t) = S_\varepsilon(t)\rho_0^{\pm,\varepsilon,per} \quad & \mp L_\varepsilon \int_0^t S_\varepsilon(t-s)\left(R_1^2R_2^2\rho^\varepsilon(s)\right) ds \\ & \mp \int_0^t S_\varepsilon(t-s)\left(\left(R_1^2R_2^2\rho^\varepsilon(s)\right)\frac{\partial \rho}{\partial x_1}^{\pm,\varepsilon,per}(s)\right) ds, \end{aligned} \tag{In_ε}$$

where $S_\varepsilon(t) = S_1(\varepsilon t)$, and $S_1(t) = e^{t\Delta}$ is a the heat semi-group, then $\rho^{\pm,\varepsilon,per}$ are solutions of the system (P_ε^{per})-(IC_ε^{per}) in the distributional sense.

For the proof of Lemma 5.4, see A. Pazy [117, Th 5.2, Page 146].

Remark 5.5
We notice that the product $(R_1^2R_2^2\rho^\varepsilon)\dfrac{\partial \rho}{\partial x_1}^{\pm,\varepsilon,per}$ is well defined in $C([0,T); L^{\frac{6}{5}}(\mathbb{T}^2))$ since $C([0,T); W^{1,\frac{3}{2}}(\mathbb{T}^2)) \hookrightarrow C([0,T); L^6(\mathbb{T}^2))$.

***Lemma* 5.6 (Time continuity)**
Let $T > 0$. If $\rho^{\pm,\varepsilon,per} \in L^\infty((0,T); W^{1,\frac{3}{2}}(\mathbb{T}^2))$ are solutions of integral problem (In_ε), then $\rho^{\pm,\varepsilon,per} \in C([0,T); W^{1,\frac{3}{2}}(\mathbb{T}^2))$.

For the proof of Lemma 5.4, see A. Pazy [117, 7.3, Page 212].

We now recall the Picard fixed point result which will be applied in Subsection 5.2 to the space $E = \left(L^\infty((0,T); W^{1,\frac{3}{2}}(\mathbb{T}^2))\right)^2$ in order to prove, the existence of solutions.

***Lemma* 5.7 (Picard Fixed point Theorem)**
Let E be a Banach space, B is a continuous bilinear application over $E \times E$ having values in E, and A a continuous linear application over E having values in E such that :

$$\|B(x,y)\|_E \leq \eta \|x\|_E \|y\|_E \quad \textit{for all} \quad x, y \in E,$$

$$\|A(x)\|_E \leq \mu \|x\|_E \quad \textit{for all} \quad x \in E,$$

where $\eta > 0$ and $\mu \in (0,1)$ are two given constants. Then, for every $x_0 \in E$ verifying

$$\|x_0\|_E < \frac{1}{4\eta}(1-\mu)^2,$$

the equation $x = x_0 + B(x,x) + A(x)$ admits a solution in E.

For the proof of Lemma 5.7, see M. Cannone [24, Lemma 4.2.14].

***Lemma* 5.8 (Decay estimate)**
Let $r, p, q \geq 1$. Then, for all functions $f \in L^q(\mathbb{T}^2)$ and $g \in L^p(\mathbb{T}^2)$, where $\frac{1}{r} \leq \frac{1}{q} + \frac{1}{p}$, we have, for $S_1(t) = e^{t\Delta}$, the following estimates :

i) $\|S_1(t)(fg)\|_{L^r(\mathbb{T}^2)} \leq Ct^{-\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)} \|f\|_{L^q(\mathbb{T}^2)} \|g\|_{L^p(\mathbb{T}^2)}$ *for all $t > 0$,*

ii) $\|\nabla S_1(t)(fg)\|_{L^r(\mathbb{T}^2)} \leq Ct^{-\left(\frac{1}{2}+\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)} \|f\|_{L^q(\mathbb{T}^2)} \|g\|_{L^p(\mathbb{T}^2)}$ *for all $t > 0$,*

with $C = C(r,p,q)$ is a positive constant.

Proof of Lemma 5.8 :

In the special case where $q = +\infty$ and $f = 1$, these estimates are the classical version of the L^r-L^p estimates for the heat semi-group (see A. Pazy [117, Lemma 1.1.8, Th 6.4.5]). The statement of Lemma 5.8 then follows by Hölder inequality :

$$\|fg\|_{L^s(\mathbb{T}^2)} \leq \|f\|_{L^q(\mathbb{T}^2)} \|g\|_{L^p(\mathbb{T}^2)}$$

with $\frac{1}{s} = \frac{1}{q} + \frac{1}{p}$. □

Here is now, the demonstration of Theorem 5.3.

5.2 Proof of Theorem 5.3

We rewrite the system (In_ε) in the following vectorial form :

$$\rho_v^\varepsilon(x,t) = S_\varepsilon(t)\rho_{0,v}^\varepsilon + L_\varepsilon \bar{J}_1 \int_0^t S_\varepsilon(t-s)\left(R_1^2R_2^2\rho^\varepsilon(s)\right)ds + \bar{I}_1\int_0^t \left(R_1^2R_2^2\rho^\varepsilon(s)\right)\frac{\partial \rho_v^\varepsilon}{\partial x_1}(s)ds,$$

where $S_\varepsilon(t) = S_1(\varepsilon t)$, $\rho_v^\varepsilon = (\rho^{+,\varepsilon,per}, \rho^{-,\varepsilon,per})$, $\rho_{0,v}^\varepsilon = (\rho_0^{+,\varepsilon,per}, \rho_0^{-,\varepsilon,per})$,
$\bar{I}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\bar{J}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

which is equivalent to,

$$\rho_v^\varepsilon(x,t) = S_\varepsilon(t)\rho_{0,v}^\varepsilon + B(\rho_v^\varepsilon, \rho_v^\varepsilon)(t) + A(\rho_v^\varepsilon)(t), \tag{5.13}$$

where B is a bilinear map and A is a linear one defined respectively, for every vector $u = (u_1, u_2)$ and $v = (v_1, v_2)$, as follows :

$$B(u,v)(t) = \bar{I}_1 \int_0^t S_\varepsilon(t-s)\left(\left(R_1^2R_2^2(u_1-u_2)\right)\frac{\partial v}{\partial x_1}(s)\right)ds, \tag{5.14}$$

$$A(u)(t) = L_\varepsilon \bar{J}_1 \int_0^t S_\varepsilon(t-s)\left(R_1^2R_2^2(u_1-u_2)(s)\right)ds. \tag{5.15}$$

Now, we apply Lemma 5.7 to equation (5.13). First of all, we estimate the bilinear term,

$$\begin{aligned}\|B(u,v)(t)\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2} &\leq \left\|\bar{I}_1\int_0^t S_\varepsilon(t-s)\left(\left(R_1^2R_2^2(u_1-u_2)\right)\frac{\partial v}{\partial x_1}(s)\right)ds\right\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2} \\ &\leq \int_0^t \left\|S_\varepsilon(t-s)\left(\left(R_1^2R_2^2(u_1-u_2)\right)\frac{\partial v}{\partial x_1}(s)\right)ds\right\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2}.\end{aligned}$$

Then, since $W^{1,\frac{3}{2}}(\mathbb{T}^2) \hookrightarrow L^4(\mathbb{T}^2)$, we have,

$$\begin{aligned}\|B(u,v)(t)\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2} \leq &\int_0^t \left\|S_\varepsilon(t-s)\left(\left(R_1^2R_2^2(u_1-u_2)\right)\frac{\partial v}{\partial x_1}(s)\right)ds\right\|_{(L^4(\mathbb{T}^2))^2} \\ &+\int_0^t \left\|\nabla S_\varepsilon(t-s)\left(\left(R_1^2R_2^2(u_1-u_2)\right)\frac{\partial v}{\partial x_1}(s)\right)ds\right\|_{(L^{\frac{3}{2}}(\mathbb{T}^2))^2}.\end{aligned} \tag{5.16}$$

We use Lemma 5.8 (i) with $r = 4, q = 3, p = \frac{3}{2}$ to estimate the first term and Lemma 5.8 (ii) with $r = \frac{3}{2}, q = 4, p = \frac{3}{2}$ to estimate the second term. We get for $0 \leq t \leq T$, and with constants C depending on ε,

$$\|B(u,v)(t)\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2} \leq C \int_0^t \frac{1}{(t-s)^{\frac{3}{4}}} \left\|R_1^2 R_2^2 u(s)\right\|_{(L^4(\mathbb{T}^2))^2} \left\|\frac{\partial v}{\partial x_1}(s)\right\|_{(L^{\frac{3}{2}}(\mathbb{T}^2))^2} ds$$

$$\leq C \sup_{0\leq s<T} (\|u(s)\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2}) \sup_{0\leq s<T} (\|v(s)\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2}) \int_0^t \frac{1}{(t-s)^{\frac{3}{4}}} ds.$$

Here we have used in the second line the property that Riesz transformations are continuous from $L^{\frac{3}{2}}$ onto itself (see A. Zygmund [139, Vol I, Page 254, (2.6)]) and the Sobolev injection. Hence we have,

$$\|B(u,v)\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)} \leq \eta(T)\|u\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)} \|v\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)}, \tag{5.17}$$

with $\eta(T) = C_0 T^{\frac{1}{4}}$ for some constant $C_0 > 0$. We estimate the linear term in the same way to get,

$$\|A(u)\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)} \leq L_\varepsilon \eta(T)\|u\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)}. \tag{5.18}$$

Moreover, we know by classical properties of heat semi-group (see A. Pazy [117]) that,

$$\|S_\varepsilon(t)\rho^\varepsilon_{0,v}\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)} \leq \|\rho^\varepsilon_{0,v}\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2}. \tag{5.19}$$

Now, if we take

$$(T^\star)^{\frac{1}{4}} = \min\left(\frac{1}{2C_0 L_\varepsilon}, \frac{1}{16 C_0 \|\rho^\varepsilon_{0,v}\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2}}\right), \tag{5.20}$$

we can easily verify that we have the following inequalities :

$$\|\rho^\varepsilon_{0,v}\|_{(W^{1,\frac{3}{2}})^2(\mathbb{T}^2)} < \frac{1}{4\eta(T^\star)}(1 - L_\varepsilon \eta(T^\star))^2, \quad \text{and } L_\varepsilon \eta(T^\star) < 1, \tag{5.21}$$

Using inequalities (5.17), (5.18), (5.19), (5.21) and Lemma 5.7 with the space $E = \left(L^\infty((0,T^\star); W^{1,\frac{3}{2}}(\mathbb{T}^2))\right)^2$, we obtain the existence of a solutions $\rho^\varepsilon_v \in \left(L^\infty((0,T^\star); W^{1,\frac{3}{2}}(\mathbb{T}^2))\right)^2$ for the system (5.13). Next, from Lemma 5.6, we deduce that $\rho^\varepsilon_v \in \left(C([0,T^\star); W^{1,\frac{3}{2}}_{loc}(\mathbb{T}^2))\right)^2$.

As a consequence, by Lemma 5.4 we prove that the system (P_ε)-(IC_ε) admits some solutions $\rho^{\pm,\varepsilon} \in C([0,T^\star); W^{1,\frac{3}{2}}_{loc}(\mathbb{R}^2))$, satisfying $(H1)$ and $(H2)$ a.e. $t \in [0,T^\star)$. $\square$

6 Properties of the solutions of (P_ε)-(IC_ε)

In this section, we are going to prove that the solutions of (P_ε)-(IC_ε) obtained by Theorem 5.3 are smooth. Moreover if we assume that the initial data (IC) satisfies $(H3)$, then the solutions are increasing in x_1 for all $t \in (0, T^\star)$.

***Lemma* 6.1 (Smoothness of the solution)**
Let $T > 0$. For all initial data $\rho_0^\pm \in L^2_{loc}(\mathbb{R}^2)$ satisfying $(H1)$ and $(H2)$, if $\rho^{\pm,\varepsilon} \in C([0,T); W^{1,\frac{3}{2}}_{loc}(\mathbb{R}^2))$ are solutions of the system (P_ε)-(IC_ε), then $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0,T))$.

Proof of Lemma 6.1 :
We denote the second term of the system (P_ε^{per}) by,

$$f^{\pm,\varepsilon} = \mp(R_1^2 R_2^2 \rho^\varepsilon)\frac{\partial \rho}{\partial x_1}^{\pm,\varepsilon,per} \mp L_\varepsilon(R_1^2 R_2^2 \rho^\varepsilon).$$

Since $W^{1,\frac{3}{2}}(\mathbb{T}^2) \hookrightarrow L^6(\mathbb{T}^2)$, $f^{\pm,\varepsilon} \in L^{\frac{6}{5}}(\mathbb{T}^2 \times (0,T))$. Moreover, we know that $\rho_0^{\pm,\varepsilon,per} \in C^\infty(\mathbb{T}^2)$. We apply the L^p regularity for the heat equation to the system (P_ε^{per})-(IC_ε^{per}), see J. L. Lions, E. Magenes [105, Th 8.2], and deduce that

$$\frac{\partial \rho}{\partial t}^{\pm,\varepsilon,per}, \quad \frac{\partial \rho}{\partial x_i}^{\pm,\varepsilon,per}, \quad \frac{\partial^2 \rho}{\partial x_i \partial x_j}^{\pm,\varepsilon,per} \in L^{\frac{6}{5}}(\mathbb{T}^2 \times (0,T)) \text{ for } \{i,j = 1,2\}.$$

We infer now by Sobolev embedding that $f^{\pm,\varepsilon} \in L^{\frac{3}{2}}(\mathbb{T}^2 \times (0,T))$. We can then iterate the previous argument with a better integrability of $f^{\pm,\varepsilon}$. By bootstrap it follows that $\rho^{\pm,\varepsilon,per} \in C^\infty(\mathbb{T}^2 \times [0,T))$. □

***Lemma* 6.2 (Strong monotonicity of the solution in x_1)**
Let $T > 0$. For all initial data $\rho_0^\pm \in L^2_{loc}(\mathbb{R}^2)$ satisfying $(H1)$, $(H2)$ and $(H3)$, if $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0,T))$ are solutions of system (P_ε)-(IC_ε), then $\dfrac{\partial \rho^{\pm,\varepsilon}}{\partial x_1} > 0$ for all $t \in (0,T)$.

Proof of Lemma 6.2 :
First of all, remark that if $\dfrac{\partial \rho_0^\pm}{\partial x_1} \geq 0$, then $\dfrac{\partial \rho_0^{\pm,\varepsilon}}{\partial x_1} \geq \varepsilon$. Indeed, we have

$$\begin{aligned}\frac{\partial \rho_0^{\pm,\varepsilon}}{\partial x_1} = \frac{\partial \rho_0^{\pm,per}}{\partial x_1} * \eta_\varepsilon + L_\varepsilon &= \left(\frac{\partial \rho_0^{\pm,per}}{\partial x_1} + L\right) * \eta_\varepsilon + \varepsilon \\ &= \left(\frac{\partial \rho_0^{\pm}}{\partial x_1}\right) * \eta_\varepsilon + \varepsilon > 0,\end{aligned}$$

since η_ε is non-negative. Let us write the system obtained by derivation of (P_ε)-(IC_ε) with respect to x_1, that reads for $\theta^{\pm,\varepsilon} = \dfrac{\partial \rho^{\pm,\varepsilon}}{\partial x_1}$,

$$\begin{cases} \dfrac{\partial \theta^{\pm,\varepsilon}}{\partial t} - \varepsilon \Delta \theta^{\pm,\varepsilon} \pm (R_1^2 R_2^2 \rho^\varepsilon) \dfrac{\partial \theta^{\pm,\varepsilon}}{\partial x_1} \pm \left(R_1^2 R_2^2 (\theta^{+,\varepsilon} - \theta^{-,\varepsilon})\right) \theta^{\pm,\varepsilon} = 0 \quad \text{in} \quad \mathbb{T}^2 \times (0,T), \\ \theta^{\pm,\varepsilon}(x,0) = \dfrac{\partial \rho_0^{\pm,\varepsilon}}{\partial x_1} \quad \text{in} \quad \mathbb{T}^2. \end{cases}$$

Since $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0,T))$ and $\theta^{\pm,\varepsilon}(\cdot,0) > \varepsilon$, we deduce from the maximum principle for scalar parabolic equations (see G. Lieberman [104, Th 2.10]), that $\theta^{\pm,\varepsilon} > 0$ on $\mathbb{T}^2 \times (0,T)$. □

Corollary 6.3 (Short time existence of monotone smooth solutions)
For all initial data $\rho_0^\pm \in L^2_{loc}(\mathbb{R}^2)$ satisfying $(H1)$, $(H2)$ and $(H3)$, and all $\varepsilon > 0$, there exists

$$T^\star(\|\rho_0^{\pm,\varepsilon,per}\|_{W^{1,\frac{3}{2}}(\mathbb{T}^2)}, L, \varepsilon) > 0,$$

such that the system (P_ε)-(IC_ε) admits solutions $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0,T^\star))$ verifying $(H1)$, $(H2)$ for all $t \in [0,T^\star)$. Moreover $\dfrac{\partial \rho^{\pm,\varepsilon}}{\partial x_1} > 0$ for all $t \in [0,T^\star)$.

Corollary 6.3 is a consequence of Theorem 5.3 and of Lemmata 6.1 and 6.2.

7 ε-Uniform estimates on the solution of the regularized system

In this Section, we prove some fundamental ε-uniform estimates. In the Subsection 7.2 we give some general estimates independent on the system of equations. In the second Subsection 7.3 we establish a priori estimates on the solutions of system (P_ε).

7.1 Properties of Hardy spaces

Definition 7.1

i) **Hardy space,** *(C. Fefferman, E. M. Stein [54]) :*

The Hardy space $\mathcal{H}^1(\mathbb{T}^2)$ is the set of functions $f \in L^1(\mathbb{T}^2)$ such that $R_i f \in L^1(\mathbb{T}^2)$ for $i = 1,2$. This space is endowed with the norm

$$\|f\|_{\mathcal{H}^1(\mathbb{T}^2)} = \|f\|_{L^1(\mathbb{T}^2)} + \sum_{i=1,2} \|R_i f\|_{L^1(\mathbb{T}^2)}.$$

ii) <u>*BMO*</u> **space,** *(John and Nirenberg, see C. Fefferman [53]) :*

We say that $f \in L^1(\mathbb{T}^2)$ belongs to $BMO(\mathbb{T}^2)$ if an only if

$$\|f\|_{BMO} = \sup_B \left(\frac{1}{|B|} \int_B |f(x) - m_B(f)| dx \right) < +\infty \tag{7.22}$$

for every ball $B \subset \mathbb{T}^2$ where $m_B(f) = \dfrac{1}{|B|} \displaystyle\int_B f$.

Here $\|f\|_{BMO}$ defines a norm over $BMO(\mathbb{T}^2)$ quotiented by the constant functions. Moreover, the space $BMO(\mathbb{T}^2)$ is the dual of $\mathcal{H}^1(\mathbb{T}^2)$.

We refer to P. Koosis [90], R. Coifman, Y. Meyer [33], J. B. Garnett [60] and E. M. Stein [126] for other definitions of $\mathcal{H}^1(\mathbb{T}^2)$ and $BMO(\mathbb{T}^2)$. Here, this definition makes a sense thanks to the definition of the Riesz transform for L^p function, and the density in L^1 of the spaces L^p for $p > 1$.
The spaces $\mathcal{H}^1$ and BMO satisfy the following properties :

***Lemma* 7.2 (Stability of Riesz transform)**

(I1) The Riesz transforms R_i, for $i = 1, 2$, are linear continuous operators on $\mathcal{H}^1(\mathbb{T}^2)$ onto itself.

(I2) The Riesz transforms R_i, for $i = 1, 2$, are linear continuous operators on $BMO(\mathbb{T}^2)$ onto itself.

(I3) The Riesz transforms R_i, for $i = 1, 2$, are linear continuous operators on $L^p(\mathbb{T}^2)$, for all $1 < p < +\infty$ onto itself.

For the proof, see R. Coifman, Y. Meyer [33, Chap 5] and A. Zygmund [139, Vol I, Page 254, (2.6)].

***Lemma* 7.3 (Embeddings)**
For $1 < p < +\infty$, we have the following property :

$$L^\infty(\mathbb{T}^2) \hookrightarrow BMO(\mathbb{T}^2) \hookrightarrow EXP(\mathbb{T}^2) \hookrightarrow L^p(\mathbb{T}^2) \hookrightarrow L\log L(\mathbb{T}^2) \hookrightarrow \mathcal{H}^1(\mathbb{T}^2) \hookrightarrow L^1(\mathbb{T}^2).$$

For the proof, see C. Bennett and R. Sharpley [16, (7.22) Page 382 , (6.11) Page 247].

***Lemma* 7.4 (Zygmund's Lemma)**
If $f \geq 0$, then $f \in L\log L(\mathbb{T}^2)$ if and only if $f \in \mathcal{H}^1(\mathbb{T}^2)$. Moreover, there exists a constant C such that,

$$\|f\|_{\mathcal{H}^1(\mathbb{T}^2)} \leq C\left(\int_{\mathbb{T}^2} |f|\log(e+|f|)dx_1dx_2 + 1\right).$$

For the proof, see A. Zygmund [139, Vol. I, Chap 7, (2.8), (2.10)] and P. Koosis [90, Page 96-97]. See also, E. M. Stein [126, 5.3, Page 128] for a proof on $\mathbb{R}^N$. Under the assumptions $(H1)$, $(H2)$, $(H3)$, and $(H4)$, we deduce that $\dfrac{\partial {\rho_0}^{\pm}}{\partial x_1} \in \mathcal{H}^1(\mathbb{T}^2)$.

7.2 Useful estimates

***Lemma* 7.5 (BMO estimate)**
If f is a function defined on $\mathbb{R}^2 \times (0,T)$ and verifies $(H1)$, $(H2)$ and $(H3)$ for a.e. $t \in (0,T)$, then there exists a constant $C = C(L)$ such that,

$$\|R_1R_2f^{per}\|_{L^\infty((0,T);BMO(\mathbb{T}^2))} \leq C,$$

where $f^{per} = f - Lx_1$.

Proof of Lemma 7.5 :
According to $(H1)$ and $(H3)$, we know that for a.e (x_2,t)

$$\int_0^1 \left|\frac{\partial f^{per}}{\partial, x_1}\right| dx_1 \leq \int_0^1 \left|\frac{\partial f}{\partial x_1} - L\right| dx_1 \leq \int_0^1 \left|\frac{\partial f}{\partial x_1}\right| dx_1 + L \leq 2L.$$

We apply a "Poincaré-Wirtinger inequality" in x_1 and we deduce that there exists a constant $C = C(L)$ such that,

$$\left\|f^{per} - \int_0^1 f^{per}dx_1\right\|_{L^\infty(\mathbb{T}^2\times(0,T))} \leq C. \tag{7.23}$$

Moreover, $R_1R_2(f^{per} - \int_0^1 f^{per}dx_1) = R_1R_2(f^{per})$ since, we can check that $R_1\left(\int_0^1 f^{per}dx_1\right) = 0$. We use Lemmata 7.3 and 7.2 $(I2)$ to obtain that $R_1R_2f^{per} \in L^\infty((0,T);BMO(\mathbb{T}^2))$.□

***Lemma* 7.6 ($L\log L$ Estimate)**
Let $(\eta_\varepsilon)_\varepsilon$ be a non-negative mollifier, then for all $f \in L\log L(\mathbb{T}^2)$, the function $f_\varepsilon = f * \eta_\varepsilon$ satisfies

$$\|f - f_\varepsilon\|_{L\log L(\mathbb{T}^2)} \to 0 \qquad as \qquad \varepsilon \to 0.$$

For the proof see R. A. Adams [2, Th 8.20].

7.3 *A priori* estimates

In this Subsection, we show some ε-uniform estimates on the solutions of the system (P_ε)-(IC_ε) obtained in Corollary 6.3. These estimates will be used, on one hand in Section 8 for the proof of long time existence, and on the other hand, in Subsection 9.2 for ensuring, by compactness, the passage to the limit as ε tends to zero.

The first estimate concerns the physical entropy of the system, and is a key result. It shows that in our model, the dislocations cannot be so concentrated. In other words, the dislocation densities can always be controlled.

***Lemma* 7.7 (Entropy estimate)**
Let $\rho_0^\pm \in L^2_{loc}(\mathbb{R}^2)$. Under the assumptions $(H1)$, $(H2)$, $(H3)$ and $(H4)$, if $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0,T))$ are solutions of the system (P_ε)-(IC_ε), then there exists a constant C independent of ε such that,

$$\left\| \frac{\partial \rho^{\pm,\varepsilon}}{\partial x_1} \right\|_{L^\infty((0,T);L\log L(\mathbb{T}^2))} + \left\| \frac{\partial}{\partial x_1} \left(R_1 R_2 \rho^\varepsilon\right) \right\|_{L^2(\mathbb{T}^2\times(0,T))} \leq C, \tag{7.24}$$

with $C = C\left(\left\| \frac{\partial \rho_0{}^\pm}{\partial x_1} \right\|_{L\log L(\mathbb{T}^2)} \right)$.

Proof of Lemma 7.7 :

First of all, we denote $\theta^{\pm,\varepsilon} = \dfrac{\partial \rho^{\pm,\varepsilon}}{\partial x_1}$, $\theta^\varepsilon = \theta^{+,\varepsilon} - \theta^{-,\varepsilon}$ and

$$N^\pm(t) = \int_{\mathbb{T}^2} \theta^{\pm,\varepsilon}(t) \log(\theta^{\pm,\varepsilon}(t)).$$

Using the fact that $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0,T))$, we can derive $N(t) = N^+(t) + N^-(t)$

with respect to t, since $\theta^{\pm,\varepsilon} > 0$ (see Lemma 6.2), and we obtain :

$$\begin{aligned}
\frac{d}{dt}N(t) &= \int_{\mathbb{T}^2} \sum_{+,-} (\theta^{\pm,\varepsilon})_t \log(\theta^{\pm,\varepsilon}) + \int_{\mathbb{T}^2} \sum_{+,-} (\theta^{\pm,\varepsilon})_t \\
&= \int_{\mathbb{T}^2} \sum_{+,-} \left(\mp(R_1^2 R_2^2 \rho^\varepsilon)\theta^{\pm,\varepsilon} + \varepsilon\Delta\rho^{\pm,\varepsilon}\right)_{x_1} \log(\theta^{\pm,\varepsilon}) \\
&= \int_{\mathbb{T}^2} \sum_{+,-} \left(\left(\pm(R_1^2 R_2^2 \rho^\varepsilon)\theta^{\pm,\varepsilon}\right) \frac{\theta^{\pm,\varepsilon}_{x_1}}{\theta^{\pm,\varepsilon}} + \varepsilon\Delta\theta^{\pm,\varepsilon} \log(\theta^{\pm,\varepsilon}) \right) \\
&= -\int_{\mathbb{T}^2} \sum_{+,-} \left(\pm(R_1^2 R_2^2 \theta^\varepsilon)\theta^{\pm,\varepsilon}\right) - \varepsilon \sum_{+,-} \int_{\mathbb{T}^2} \frac{|\nabla\theta^{\pm,\varepsilon}|^2}{\theta^{\pm,\varepsilon}} \\
&= -\int_{\mathbb{T}^2} \left(R_1^2 R_2^2 \theta^\varepsilon\right) \theta^\varepsilon - \varepsilon \sum_{+,-} \int_{\mathbb{T}^2} \frac{|\nabla\theta^{\pm,\varepsilon}|^2}{\theta^{\pm,\varepsilon}} \\
&= -\int_{\mathbb{T}^2} (R_1 R_2 \theta^\varepsilon)^2 - \varepsilon \sum_{+,-} \int_{\mathbb{T}^2} \frac{|\nabla\theta^{\pm,\varepsilon}|^2}{\theta^{\pm,\varepsilon}} \leq 0,
\end{aligned}$$

Integrating in time we get,

$$N(t) + \int_0^t \int_{\mathbb{T}^2} (R_1 R_2 \theta^\varepsilon)^2 \leq N(0) \leq \int_{\mathbb{T}^2} \sum_{+,-} \theta^{\pm,\varepsilon}(0) \log(e + \theta^{\pm,\varepsilon}(0))$$

Since the initial data (IC) satisfies $(H4)$, we deduce by Lemma 7.6 that there exists a positive constant C independent of ε such that,

$$N(t) + \int_0^t \int_{\mathbb{T}^2} (R_1 R_2 \theta^\varepsilon)^2 \leq C.$$

Let us now consider,

$$\begin{aligned}
N_1^{\pm}(t) &= \int_{\mathbb{T}^2} \theta^{\pm,\varepsilon}(t) \log(e + \theta^{\pm,\varepsilon}(t)) \\
&= \int_{\mathbb{T}^2 \cap \{0 < \theta^{\pm,\varepsilon} < e\}} \theta^{\pm,\varepsilon}(t) \log(e + \theta^{\pm,\varepsilon}(t)) + \int_{\mathbb{T}^2 \cap \{\theta^{\pm,\varepsilon} \geq e\}} \theta^{\pm,\varepsilon}(t) \log(e + \theta^{\pm,\varepsilon}(t)).
\end{aligned}$$

Using that $x \log(e + x) \leq e \log(2e)$ for all $0 < x \leq e$, we deduce that

$$
\begin{aligned}
N_1^{\pm}(t) &\leq e\log(2e) + \int_{\mathbb{T}^2\cap\{\theta^{\pm,\varepsilon}\geq e\}} \theta^{\pm,\varepsilon}(t)\log(2\theta^{\pm,\varepsilon}(t)) \\
&\leq e\log(2e) + \int_{\mathbb{T}^2} \theta^{\pm,\varepsilon}(t)\log(2) + \int_{\mathbb{T}^2\cap\{\theta^{\pm,\varepsilon}\geq e\}} \theta^{\pm,\varepsilon}(t)\log(\theta^{\pm,\varepsilon}(t)) \\
&\leq e\log(2e) + \log(2)(L+1) + N^{\pm}(t) - \int_{\mathbb{T}^2\cap\{0<\theta^{\pm,\varepsilon}<e\}} \theta^{\pm,\varepsilon}(t)\log(\theta^{\pm,\varepsilon}(t)) \\
&\leq C + N^{\pm}(t),
\end{aligned}
$$

where, in the last line we have used that $-x\log(x) \leq \frac{1}{e}$ for all $0 < x \leq e$. This finally lead to the following estimate :

$$
N_1^{+}(t) + N_1^{-}(t) + \int_0^t \int_{\mathbb{T}^2} \left(R_1R_2\theta^{\varepsilon}\right)^2 \leq C,
$$

which implies (7.24). □

Remark 7.8 **($W^{1,2}$ estimate)**
Since we have

$$
\frac{\partial}{\partial x_2} R_1^2R_2^2 = R_1R_2\left(\frac{\partial}{\partial x_1}R_1R_2\right) \quad \textit{and} \quad \frac{\partial}{\partial x_2} R_1^2R_2^2 = R_2^2\left(\frac{\partial}{\partial x_1}R_1R_2\right),
$$

we deduce by Lemma 7.2 (I3), that $\nabla\left(R_1^2R_2^2\rho^{\varepsilon}\right) \in L^2\left(\mathbb{T}^2\times(0,T)\right)$ *uniformly in* ε.

Remark 7.9 **($\mathcal{H}^1$ estimate)**
Given $\theta^{\pm,\varepsilon} \geq 0$, *we deduce from Lemma 7.4 that* $\theta^{\pm,\varepsilon} \in L^{\infty}((0,T);\mathcal{H}^1(\mathbb{T}^2))$, *uniformly in* ε.

We now present a second a priori estimate.

***Lemma* 7.10** **(L^2 bound on the solutions)**
Let $T > 0$. *Under the condition* $\rho_0^{\pm} \in L^2_{loc}(\mathbb{R}^2)$, *and the assumptions* $(H1)$, $(H2)$, $(H3)$ *and* $(H4)$, *if* $\rho^{\pm,\varepsilon} \in C^{\infty}(\mathbb{R}^2\times[0,T))$ *are solutions of system* (P_ε)-(IC_ε), *then there exists a constant* C *independent of* ε, *but depending on* T, *such that :*

$$
\left\|\rho^{\pm,\varepsilon,per}\right\|_{L^{\infty}((0,T);L^2(\mathbb{T}^2))} \leq C,
$$

with $\rho^{\pm,\varepsilon,per} = \rho^{\pm,\varepsilon} - Lx_1$.

Proof of Lemma 7.10 :
We want to bound $m^{\pm,\varepsilon}(x_2,t) = \int_{\mathbb{T}} \rho^{\pm,\varepsilon,per}(x_1,x_2,t)dx_1$. There is no problem of regularity since $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0,T))$. We integrate equation (P_ε^{per}) with respect to x_1, and then integrate by parts the first term of the right hand side. This leads to,

$$\begin{aligned}\frac{\partial}{\partial t}m^{\pm,\varepsilon} - \varepsilon\frac{\partial^2 m^{\pm,\varepsilon}}{\partial x_2^2} = & \pm\int_{\mathbb{T}}(R_1^2R_2^2\frac{\partial\rho^\varepsilon}{\partial x_1})(\rho^{\pm,\varepsilon,per} - m^{\pm,\varepsilon})dx_1 \\ & \mp L_\varepsilon\int_{\mathbb{T}}(R_1^2R_2^2\rho^\varepsilon)dx_1 \pm m^{\pm,\varepsilon}\int_{\mathbb{T}}(R_1^2R_2^2\frac{\partial\rho^\varepsilon}{\partial x_1})dx_1.\end{aligned}$$

Using that ρ^ε is a 1-periodic function in x_1, the previous equation is equivalent to,

$$\frac{\partial}{\partial t}m^{\pm,\varepsilon} - \varepsilon\frac{\partial^2 m^{\pm,\varepsilon}}{\partial x_2^2} = \overbrace{\pm\int_{\mathbb{T}}(R_1^2R_2^2\frac{\partial\rho^\varepsilon}{\partial x_1})(\rho^{\pm,\varepsilon,per} - m^{\pm,\varepsilon})dx_1}^{I_1^\pm} \overbrace{\mp L_\varepsilon\int_{\mathbb{T}}(R_1^2R_2^2\rho^\varepsilon)dx_1}^{I_2^\mp}. \tag{7.25}$$

Let us denote the right hand side by $g^\pm = I_1^\pm + I_2^\mp$. We now show that $g^\pm \in L^2(\mathbb{T}\times(0,T))$. Indeed, we have,

$$\begin{aligned}\|I_1^\pm\|_{L^2(\mathbb{T}\times(0,T))} &\leq \left\|\int_{\mathbb{T}}(R_1^2R_2^2\frac{\partial\rho^\varepsilon}{\partial x_1})(\rho^{\pm,\varepsilon,per} - m^{\pm,\varepsilon})dx_1\right\|_{L^2(\mathbb{T}\times(0,T))} \\ &\leq \|\rho^{\pm,\varepsilon,per} - m^{\pm,\varepsilon}\|_{L^\infty(\mathbb{T}^2\times(0,T))}\left\|R_1^2R_2^2\frac{\partial\rho^\varepsilon}{\partial x_1}\right\|_{L^2(\mathbb{T}^2\times(0,T))} \\ &\leq C,\end{aligned}$$

where for the last line we used Lemma 7.7 to bound $\left\|R_1^2R_2^2\frac{\partial\rho^\varepsilon}{\partial x_1}\right\|_{L^2(\mathbb{T}^2\times(0,T))}$ and the fact that the Riesz transforms are continuous from L^2 onto itself. Furthermore, the bound on $\|\rho^{\pm,\varepsilon,per} - m^{\pm,\varepsilon}\|_{L^\infty(\mathbb{T}^2\times(0,T))}$ follows from (7.23).

For the term $I_2^\mp$, recall that $0 < \varepsilon \leq 1$, hence

$$\|I_2^\mp\|_{L^2(\mathbb{T}\times(0,T))} \leq \left\|(L+1)\int_{\mathbb{T}}(R_1^2R_2^2\rho^\varepsilon)dx_1\right\|_{L^2(\mathbb{T}\times(0,T))} \leq CT^{\frac{1}{2}},$$

where for the last inequality we have used that $R_1^2R_2^2\rho^\varepsilon \in L^\infty((0,T);BMO(\mathbb{T}^2))$ (see Lemma 7.5) and the embeddings of Lemma 7.3. Therefore, we get,

$$\|g^\pm\|_{L^2(\mathbb{T}\times(0,T))} \leq C(1+T^{\frac{1}{2}}).$$

To end the proof, we multiply equation (7.25) by $m^{\pm,\varepsilon}$, and we integrate in space. This gives,

$$\frac{1}{2}\frac{d}{dt}\|m^{\pm,\varepsilon}(t)\|^2_{L^2(\mathbb{T})} + \varepsilon\left\|\frac{\partial}{\partial x_2}m^{\pm,\varepsilon}(t)\right\|^2_{L^2(\mathbb{T})} = \int_{\mathbb{T}} g^{\pm}m^{\pm,\varepsilon}.$$

We integrate in time, to obtain,

$$\begin{aligned}
\frac{1}{2}\|m^{\pm,\varepsilon}\|^2_{L^\infty((0,T);L^2(\mathbb{T}))} &\leq \|g^{\pm}\|_{L^2(\mathbb{T}\times(0,T))}\|m^{\pm,\varepsilon}\|_{L^2(\mathbb{T}\times(0,T))} + \frac{1}{2}\|m^{\pm,\varepsilon}(0)\|^2_{L^2(\mathbb{T})} \\
&\leq T^{\frac{1}{2}}\|g^{\pm}\|_{L^2(\mathbb{T}\times(0,T))}\|m^{\pm,\varepsilon}\|_{L^\infty((0,T);L^2(\mathbb{T}))} + \frac{1}{2}\|m^{\pm,\varepsilon}(0)\|^2_{L^2(\mathbb{T})} \\
&\leq T\|g^{\pm}\|^2_{L^2(\mathbb{T}\times(0,T))} + \frac{1}{4}\|m^{\pm,\varepsilon}\|^2_{L^\infty((0,T);L^2(\mathbb{T}))} + \frac{1}{2}\|m^{\pm,\varepsilon}(0)\|^2_{L^2(\mathbb{T})}.
\end{aligned}$$

Therefore

$$\|m^{\pm,\varepsilon}\|^2_{L^\infty((0,T);L^2(\mathbb{T}))} \leq 4T\|g^{\pm}\|^2_{L^2(\mathbb{T}\times(0,T))} + 2\|m^{\pm,\varepsilon}(0)\|^2_{L^2(\mathbb{T})}.$$

We now bound the term $\|m^{\pm,\varepsilon}(0)\|^2_{L^2(\mathbb{T})}$. We have,

$$\begin{aligned}
\|m^{\pm,\varepsilon}(0)\|^2_{L^2(\mathbb{T})} &= \int_{\mathbb{T}}\left|\int_{\mathbb{T}}\rho_0^{\pm,\varepsilon,per}(x_1,x_2)dx_1\right|^2 dx_2 \\
&\leq \|\eta_\varepsilon * \rho_0^{\pm,per}\|^2_{L^2(\mathbb{T}^2)} \\
&\leq \|\rho_0^{\pm,per}\|^2_{L^2(\mathbb{T}^2)},
\end{aligned}$$

where we have used Hölder's inequality for the second line, and that $\|\eta_\varepsilon\|_{L^1(\mathbb{T}^2)} = 1$. This indicates that for a constant C independent of ε, $\|m^{\pm,\varepsilon}\|_{L^\infty((0,T);L^2(\mathbb{T}))} \leq C$.

Finally, we use estimate (7.23) to deduce that $\rho^{\pm,\varepsilon,per}$ is bounded in $L^\infty((0,T);L^2(\mathbb{T}^2))$ uniformly in ε. □

The following estimate will provide compactness in time of the solution, uniform with respect to ε .

***Lemma* 7.11 (Duality estimate of Riesz transform for the time derivative of the solution)**
Let $T > 0$. Under the assumptions $\rho_0^{\pm} \in L^2_{loc}(\mathbb{R}^2)$, $(H1)$, $(H2)$, $(H3)$ and $(H4)$, if $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0,T))$ are solutions of the system (P_ε)-(IC_ε), then for all $\psi \in L^2((0,T);W^{1,2}(\mathbb{T}^2))$, there exists a constant C independent of ε such that :

$$\left|\int_{\mathbb{T}^2\times(0,T)} \psi R_1^2 R_2^2 \left(\frac{\partial \rho^\varepsilon}{\partial t}\right)\right| \leq C\|\psi\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))},$$

where $\rho^\varepsilon = \rho^{+,\varepsilon} - \rho^{-,\varepsilon}$.

Proof of Lemma 7.11 :

The idea is somehow to bound $R_1^2 R_2^2 \left(\frac{\partial \rho^\varepsilon}{\partial t}\right)$ using the available bounds on the right hand side of the equation (P_ε).

We will give a proof by duality. First of all, we subtract the two equations of system (P_ε) to obtain that,

$$\frac{\partial \rho^\varepsilon}{\partial t} = -(R_1^2 R_2^2 \rho^\varepsilon)\left(\frac{\partial \rho^{+\varepsilon}}{\partial x_1} + \frac{\partial \rho^{-\varepsilon}}{\partial x_1}\right) + \varepsilon \Delta \rho^{\pm,\varepsilon}.$$

We apply the Riesz transform $R_1^2 R_2^2$, which gives,

$$R_1^2 R_2^2 \left(\frac{\partial \rho^\varepsilon}{\partial t}\right) = -\overbrace{R_1^2 R_2^2 \left((R_1^2 R_2^2 \rho^\varepsilon)\frac{\partial k^\varepsilon}{\partial x_1}\right)}^{I_1} + \overbrace{\varepsilon R_1^2 R_2^2 (\Delta \rho^\varepsilon)}^{I_2}, \tag{7.26}$$

with $k^\varepsilon = \rho^{+,\varepsilon} + \rho^{-,\varepsilon}$. In what follows, we will prove that for a function $\psi \in L^2((0,T);W^{1,2}(\mathbb{T}^2))$, we can bound $J_i = \int_{\mathbb{T}^2\times(0,T)} \psi I_i$ for $i = 1, 2$.

<u>**Estimate of** J_1</u> : to control J_1, we rewrite it under the following form :

$$\int_{\mathbb{T}^2\times(0,T)} R_1^2 R_2^2 \left((R_1^2 R_2^2 \rho^\varepsilon)\frac{\partial k^\varepsilon}{\partial x_1}\right)\psi = \int_{\mathbb{T}^2\times(0,T)} \left((R_1^2 R_2^2 \rho^\varepsilon)\frac{\partial k^\varepsilon}{\partial x_1}\right) R_1^2 R_2^2(\psi).$$

We use the fact that,

(i) $(R_1^2 R_2^2 \rho^\varepsilon)$ is bounded in $L^\infty((0,T);W^{1,2}(\mathbb{T}^2))$ uniformly in ε (by Lemma 7.7),

(ii) $\frac{\partial k^\varepsilon}{\partial x_1}$ is bounded in $L^\infty((0,T);L\log L(\mathbb{T}^2))$, uniformly in ε (by Lemma 7.7).

We deduce from this and from Proposition 4.6, (with $f = R_1^2 R_2^2 \rho^\varepsilon$ and $g = \frac{\partial k^\varepsilon}{\partial x_1}$) the following estimate :

$$\begin{aligned}\left\|(R_1^2 R_2^2 \rho^\varepsilon)\frac{\partial k^\varepsilon}{\partial x_1}\right\|_{L^2((0,T);L\log^{\frac{1}{2}} L(\mathbb{T}^2))} &\leq C\|R_1^2 R_2^2 \rho^\varepsilon\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))}\left\|\frac{\partial k^\varepsilon}{\partial x_1}\right\|_{L^2((0,T);L\log L(\mathbb{T}^2))}\\ &\leq C\left\|\frac{\partial k^\varepsilon}{\partial x_1}\right\|_{L^\infty((0,T);L\log L(\mathbb{T}^2))} \leq C.\end{aligned}$$

We use Lemma 4.3 (i), to deduce that

$$\begin{aligned}
|J_1| &\leq \left| \int_{\mathbb{T}^2\times(0,T)} \left((R_1^2 R_2^2 \rho^\varepsilon) \frac{\partial k}{\partial x_1}^{\varepsilon} \right) R_1^2 R_2^2(\psi) \right| \\
&\leq \left\| (R_1^2 R_2^2 \rho^\varepsilon) \frac{\partial k}{\partial x_1}^{\varepsilon} \right\|_{L^2((0,T);L\log^{\frac{1}{2}} L(\mathbb{T}^2))} \|R_1^2 R_2^2 \psi\|_{L^2((0,T);EXP_2(\mathbb{T}^2))} \\
&\leq C \, \|R_1^2 R_2^2 \psi\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))} \\
&\leq C \, \|\psi\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))} \, ,
\end{aligned} \tag{7.27}$$

where we have used the Trudinger embedding (see Lemma 4.4) in the third line and the fact that Riesz transforms are continuous from $W^{1,2}$ onto itself in the last line.
Estimate of J_2 : to estimate J_2, we integrate by parts, to get :

$$J_2 = -\varepsilon \int_{\mathbb{T}^2\times(0,T)} \nabla(R_1^2 R_2^2 \rho^\varepsilon) \cdot \nabla\psi.$$

Since $R_1^2 R_2^2 \rho^\varepsilon$ is bounded in $L^2((0,T);W^{1,2}(\mathbb{T}^2))$, we deduce that for all $0 < \varepsilon \leq 1$:

$$\begin{aligned}
|J_2| &\leq \left| \int_{\mathbb{T}^2\times(0,T)} \nabla(R_1^2 R_2^2 \rho^\varepsilon) \cdot \nabla\psi \right| \\
&\leq C \, \|R_1^2 R_2^2 \rho^\varepsilon\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))} \, \|\psi\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))}.
\end{aligned} \tag{7.28}$$

Finally, collecting (7.27) and (7.28) together with (7.26) and the definitions of J_i, for $i = 1, 2$, we get that there exists a constant C independent of ε such that,

$$\left| \int_{\mathbb{T}^2\times(0,T)} \psi R_1^2 R_2^2 (\frac{\partial \rho^\varepsilon}{\partial t}) \right| \leq C \|\psi\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))}.$$

□

Remark 7.12 ($W^{-1,2}$ **estimate)**
Let $W^{-1,2}(\mathbb{T}^2)$ be the dual space of $W^{1,2}(\mathbb{T}^2)$. Thanks to the previous lemma we deduce that there exists a constant C independent of ε such that,

$$\left\| R_1^2 R_2^2 \left(\frac{\partial \rho^\varepsilon}{\partial t} \right) \right\|_{L^2((0,T);W^{-1,2}(\mathbb{T}^2))} \leq C.$$

These three estimates made in Lemmata 7.7, 7.10 and 7.11 are sufficient to obtain the required compactness. This compactness ensures in Subsection 9.2 the passage to the limit which allows us to show the existence of solutions.

***Lemma* 7.13 (Duality estimate for the time derivative of the solution)**
Let $T > 0$. Under the assumptions $\rho_0^{\pm} \in L^2_{loc}(\mathbb{R}^2)$, $(H1)$, $(H2)$, $(H3)$ and $(H4)$, if $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0,T))$ are solutions of the system (P_ε)-(IC_ε), then for all $\psi \in L^2((0,T); W^{2,2}(\mathbb{T}^2))$, there exists a constant C independent of ε such that,

$$\left| \int_{\mathbb{T}^2 \times (0,T)} \psi \left(\frac{\partial \rho^{\pm,\varepsilon}}{\partial t} \right) \right| \leq C \|\psi\|_{L^2((0,T);W^{2,2}(\mathbb{T}^2))}.$$

The proof of this lemma is similar to that of Lemma 7.11. The only difference is that we integrate by parts the viscosity term twice and use the estimate of Lemma 7.10.

Remark 7.14 (The sense of the initial condition)
According to this lemma and Lemma 7.10 , we have $\rho^{\pm,\varepsilon,per} \in C([0,T), W^{-2,2}(\mathbb{T}^2))$ uniformly in ε with $W^{-2,2}(\mathbb{T}^2)$ is the dual space of $W^{2,2}(\mathbb{T}^2)$. This will give later a sense to the limit of the initial conditions.

8 Global existence for the regularized system

In this Section, we will prove the global existence of solutions for the system (P_ε)-(IC_ε) using the previous a priori estimates (proven in Lemmata 7.5 and 7.7).

Before going into the proof, we need the following lemma.

***Lemma* 8.1 ($W^{1,\frac{3}{2}}$ estimate)**
For all initial data $\rho_0^{\pm} \in L^2_{loc}(\mathbb{R}^2)$ satisfying $(H1)$ and $(H2)$, if $\rho^{\pm,\varepsilon,per} \in C^\infty(\mathbb{T}^2 \times [0,T))$, are solutions of the Mild integral problem (In_ε), then there exists a constant $C = C(\varepsilon, L)$ such that,

$$\|\rho^{\pm,\varepsilon,per}\|_{L^\infty((0,T);W^{1,\frac{3}{2}}(\mathbb{T}^2))} \leq B_0^{\pm} + CT^{\frac{1}{24}} \|R_1^2 R_2^2 \rho^\varepsilon\|_{L^\infty((0,T);L^8(\mathbb{T}^2))} \left(\left\| \frac{\partial \rho^{\pm,\varepsilon}}{\partial x_1} \right\|_{L^\infty((0,T);L^1(\mathbb{T}^2))} + 1 \right),$$

where $B_0^{\pm} = \|\rho_0^{\pm,\varepsilon,per}\|_{W^{1,\frac{3}{2}}(\mathbb{T}^2)}$.

Proof of Lemma 8.1 :
If we denote $\rho_v^\varepsilon = (\rho^{+,\varepsilon,per}, \rho^{-,\varepsilon,per})$ and $\rho_{0,v}^\varepsilon = (\rho_0^{+,\varepsilon,per}, \rho_0^{-,\varepsilon,per})$, then we have shown that ρ_v^ε satisfies (5.13), namely,

$$\rho_v^\varepsilon(x,t) = S(t)\rho_{0,v}^\varepsilon + B(\rho_v^\varepsilon, \rho_v^\varepsilon)(t) + A(\rho_v^\varepsilon)(t), \tag{8.29}$$

where B and A are defined in (5.14) and (5.15) respectively and where $S_\varepsilon(t) = S_1(\varepsilon t)$. Moreover, using (5.16) with $u = v = \rho_v^\varepsilon$, we get,

$$\|B(\rho_v^\varepsilon,\rho_v^\varepsilon)(t)\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2} \leq \int_0^t \left\| S_\varepsilon(t-s)\left(\left(R_1^2R_2^2\rho^\varepsilon(s)\right)\frac{\partial\rho_v^\varepsilon}{\partial x_1}(s)\right)ds\right\|_{(L^4(\mathbb{T}^2))^2}$$
$$+\int_0^t \left\|\nabla S_\varepsilon(t-s)\left(\left(R_1^2R_2^2\rho^\varepsilon(s)\right)\frac{\partial\rho_v^\varepsilon}{\partial x_1}(s)\right)ds\right\|_{(L^{\frac{3}{2}}(\mathbb{T}^2))^2}.$$

We use now Lemma 5.8 (i) with $r=4, q=\frac{24}{5}, p=1$ to estimate the first term, and Lemma 5.8 (ii) with $r=\frac{3}{2}, q=8, p=1$ to estimate the second term. It gives for $t\in(0,T)$, that,

$$\|B(\rho_v^\varepsilon,\rho_v^\varepsilon)(t)\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2} \leq C\int_0^t \frac{1}{(t-s)^{\frac{23}{24}}}\left\|R_1^2R_2^2\rho^\varepsilon(s)\right\|_{L^8(\mathbb{T}^2)}\left\|\frac{\partial\rho_v^\varepsilon}{\partial x_1}(s)\right\|_{(L^1(\mathbb{T}^2))^2}ds$$
$$\leq C\sup_{0\leq s<T}\left(\left\|R_1^2R_2^2\rho^\varepsilon(s)\right\|_{L^8(\mathbb{T}^2)}\right)\sup_{0\leq s<T}\left(\left\|\frac{\partial\rho_v^\varepsilon}{\partial x_1}(s)\right\|_{(L^1(\mathbb{T}^2))^2}\right)\int_0^t\frac{1}{(t-s)^{\frac{23}{24}}}.$$

That leads,

$$\|B(\rho_v^\varepsilon,\rho_v^\varepsilon)\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)} \leq CT^{\frac{1}{24}}\|R_1^2R_2^2\rho^\varepsilon\|_{L^\infty((0,T);L^8(\mathbb{T}^2))}\left\|\frac{\partial\rho_v^\varepsilon}{\partial x_1}\right\|_{L^\infty((0,T);(L^1(\mathbb{T}^2))^2)}. \tag{8.30}$$

Similarly, we show that,

$$\|A(\rho_v^\varepsilon)\|_{L^\infty((0,T);W^{1,\frac{3}{2}}(\mathbb{T}^2))} \leq CT^{\frac{1}{24}}\|R_1^2R_2^2\rho^\varepsilon\|_{L^\infty((0,T);L^8(\mathbb{T}^2))}. \tag{8.31}$$

By using (8.30), (8.31) and (5.19), and the equation (8.29) we get the proof. □

Theorem 8.2 (Global existence)
For all $T,\varepsilon>0$ and for all initial data $\rho_0^\pm\in L^2_{loc}(\mathbb{R}^2)$ satisfies $(H1)$, $(H2)$, $(H3)$ and $(H4)$ the system (P_ε)-(IC_ε) admits solutions $\rho^{\pm,\varepsilon}\in C^\infty(\mathbb{R}^2\times[0,T))$. Moreover, this solution satisfies $(H1)$, $(H2)$ and $(H3)$ for all $t\in(0,T)$ and the estimates given in Lemmata 7.7, 7.10 and 7.11.

Proof of Theorem 8.2 :
In Theorem 8.2, we prove that the local solutions given by Corollary 6.3 can be extended to some global ones. We argue by contradiction. Suppose that there exists a maximum time T_{max} such that we have the existence of solutions of (P_ε)-(IC_ε) in $C^\infty(\mathbb{R}^2\times[0,T_{max}))$.

For $\delta > 0$, we reconsider the system (P_ε) with the initial data

$$\rho^{\pm,\varepsilon}_{\delta,max} = \rho^{\pm,\varepsilon}(x, T_{max} - \delta).$$

we reapply for the second time, the proof of Corollary 6.3, we deduce that there exists a time

$$T^\star_{\delta,max}(\|\rho^{\pm,\varepsilon,per}_{\delta,max}\|_{W^{1,\frac{3}{2}}(\mathbb{T}^2)}, L, \varepsilon) > 0, \quad \text{where} \quad \rho^{\pm,\varepsilon,per}_{\delta,max} = \rho^{\pm,\varepsilon}_{\delta,max} - Lx_1,$$

such that the system (P_ε)-(IC_ε) admits solutions defined until,

$$T_0 = (T_{max} - \delta) + T^\star_{\delta,max}.$$

Moreover, by Lemmata 8.1, 7.7 and 7.5, we know that $\rho^{\pm,\varepsilon,per}_{\delta,max}$ are δ-uniformly bounded in $W^{1,\frac{3}{2}}(\mathbb{T}^2)$. By using (5.20), we deduce that there exists a constant $C(\varepsilon, T_{max}, L) > 0$ independent of δ such that $T^\star_{\delta,max} \geq C > 0$. Then $\liminf_{\delta\to 0} T^\star_{\delta,max} \geq C > 0$. Hence $T_0 > T_{max}$ which gives the contradiction. □

9 Existence of solutions for the system (P)-(IC)

In this section, we will prove that the system (P)-(IC) admits solutions $\rho^\pm$ in the distributional sense. They are the limits of $\rho^{\pm,\varepsilon}$ given by Theorem 8.2 when $\varepsilon \to 0$. To do this, we will justify the passage to the limit as ε tends to 0 in the system (P^{per}_ε)-(IC^{per}_ε) by using some compactness arguments.

9.1 Preliminary results

***Lemma* 9.1 (Trudinger compact embedding)**
The following injection (see N. S. Trudinger [129]) :

$$W^{1,2}(\mathbb{T}^2) \hookrightarrow EXP_\beta(\mathbb{T}^2),$$

is compact, for all $1 \leq \beta < 2$.

For the proof of this lemma see also R. A. Adams [2, Th 8.32].

***Lemma* 9.2 (Simon's Lemma)**
Let X, B, Y three Banach spaces, where $X \hookrightarrow B$ with compact embedding and $B \hookrightarrow Y$ with continuous embedding. If $(\rho^n)_n$ is a sequence such that,

$$\|\rho^n\|_{L^q((0,T);B)} + \|\rho^n\|_{L^1((0,T);X)} + \left\|\frac{\partial \rho^n}{\partial t}\right\|_{L^1((0,T);Y)} \leq C,$$

where $q > 1$ and C is a constant independent of n, then $(\rho^n)_n$ is relatively compact in $L^p((0,T);B)$ for all $p < q$.

For the proof, see J. Simon [125, Th 6, Page 86].
In order to show the existence of system (P) in Subsection 9.2, we apply this lemma in the particular cases where $B = EXP_\beta(\mathbb{T}^2)$, $X = W^{1,2}(\mathbb{T}^2)$ and $Y = W^{-1,2}(\mathbb{T}^2)$, for $1 < \beta < 2$.

***Lemma* 9.3 (Weak star topology in $L \log L$)**
Let $E_{exp}(\mathbb{T}^2)$ be the closure in $EXP(\mathbb{T}^2)$ of the space of functions bounded on $\mathbb{T}^2$. Then $E_{exp}(\mathbb{T}^2)$ is a separable Banach space which verifies,

i) $$L \log L(\mathbb{T}^2) \text{ is the dual space of } E_{exp}(\mathbb{T}^2).$$

ii) $$EXP_\beta(\mathbb{T}^2) \hookrightarrow E_{exp}(\mathbb{T}^2) \hookrightarrow EXP(\mathbb{T}^2) \text{ for all } \beta > 1.$$

For the proof, see R. A. Adams [2, Th 8.16, 8.18, 8.20].

9.2 Proof of Theorem 1.4

Step 1 (Passage to the limit) :

First, by Lemma 7.10 we know that for any $T > 0$, the solutions $\rho^{\pm,\varepsilon,per}$ of the system (P_ε^{per})-(IC_ε^{per}) obtained with the help of Theorem 8.2, are ε-uniformly bounded in $L^2(\mathbb{T}^2 \times (0,T))$. Hence, as ε goes to zero, we can extract a subsequence still denoted by $\rho^{\pm,\varepsilon,per}$, that converges weakly in $L^2(\mathbb{T}^2 \times (0,T))$ to some limit $\rho^{\pm,per}$. Then we want to prove that $\rho^\pm = \rho^{\pm,per} + Lx_1$ are solutions of the system (P)-(IC). Indeed, since the passage to the limit in the linear term is trivial in $\mathcal{D}'(\mathbb{T}^2 \times (0,T))$, it suffices to pass to the limit in the non-linear term,

$$(R_1^2 R_2^2 \rho^\varepsilon) \frac{\partial}{\partial x_1} \rho^{\pm,\varepsilon,per}.$$

• From Lemmata 7.7 and 7.5 we know that the term $(R_1^2 R_2^2 \rho^\varepsilon)$ is ε-uniformly bounded in $L^2((0,T); W^{1,2}(\mathbb{T}^2))$. Then it is in particular ε-uniformly bounded in $L^1((0,T); W^{1,2}(\mathbb{T}^2))$.
• From thes previous point and Lemma 9.1, we know that $(R_1^2 R_2^2 \rho^\varepsilon)$ is also ε-uniformly bounded in $L^2((0,T); EXP_\beta(\mathbb{T}^2))$ for all $1 \leq \beta < 2$.
• From Lemma 7.11, the term $R_1^2 R_2^2 (\frac{\partial \rho^\varepsilon}{\partial t})$ is ε-uniformly bounded in $L^2((0,T); W^{-1,2}(\mathbb{T}^2))$ and then in $L^1((0,T); W^{-1,2}(\mathbb{T}^2))$.

Collecting this, we get that there exists a constant C independent on ε such that $\bar{\rho}^\varepsilon = R_1^2 R_2^2 \rho^\varepsilon$ satisfies for some $1 < \beta < 2$

$$\|\bar{\rho}^\varepsilon\|_{L^2((0,T);EXP_\beta(\mathbb{T}^2))} + \|\bar{\rho}^\varepsilon\|_{L^1((0,T);W^{1,2}(\mathbb{T}^2))} + \left\| \frac{\partial \bar{\rho}^\varepsilon}{\partial t} \right\|_{L^1((0,T);W^{-1,2}(\mathbb{T}^2))} \leq C.$$

Then Lemma 9.2, with $B = EXP_\beta(\mathbb{T}^2)$, $X = W^{1,2}(\mathbb{T}^2)$ and $Y = W^{-1,2}(\mathbb{T}^2)$, shows the relative compactness of $(R_1^2 R_2^2 \rho^\varepsilon)$ in $L^1((0,T); EXP_\beta(\mathbb{T}^2))$, and then using Lemma 9.3, we have the compactness in $L^1((0,T); E_{exp}(\mathbb{T}^2))$.

Moreover, by Lemma 7.7, we have that $\dfrac{\partial \rho}{\partial x_1}^{\pm,\varepsilon,per}$ is ε-uniformly bounded in $L^\infty((0,T); L\log L(\mathbb{T}^2))$ which is the dual of $L^1((0,T); E_{exp}(\mathbb{T}^2))$ by Lemma 9.3 (see T. Cazenave and A. Haraux [28, Th 1.4.19, Page 17]). Then, this final term converges weakly $\star$ in $L^\infty((0,T); L\log L(\mathbb{T}^2))$ toward $\dfrac{\partial \rho}{\partial x_1}^{\pm,per}$. That enables us to pass to the limit in the bilinear term in the sense

$$L^1((0,T); E_{exp}(\mathbb{T}^2)) - strong \ \times \ L^\infty((0,T); L\log L(\mathbb{T}^2)) - weak \star.$$

In what precedes, we have shown that $\rho^\pm = \rho^{\pm,per} + Lx_1$ are solutions of the following equation :

$$\begin{aligned}\frac{\partial \rho^\pm}{\partial t} &= \mp (R_1^2 R_2^2 \rho) \frac{\partial \rho}{\partial x_1}^{\pm,per} \mp L(R_1^2 R_2^2 \rho) \\ &= \mp (R_1^2 R_2^2 \rho) \frac{\partial \rho}{\partial x_1}^{\pm}.\end{aligned}$$

Therefore $\rho^\pm$ is solutions of system (P) which has the same bounds as $\rho^{\pm,\varepsilon}$. At this stage we remark that, by Proposition 4.6, the second term of the previous system is in $L^2((0,T); L\log^{\frac{1}{2}} L(\mathbb{T}^2))$, which gives that $\dfrac{\partial \rho^\pm}{\partial t} \in L^2((0,T); L\log^{\frac{1}{2}} L(\mathbb{T}^2))$, and then $\rho^{\pm,per} \in C([0,T); L\log^{\frac{1}{2}} L(\mathbb{T}^2))$.

Step 2 (The initial conditions) :

It remains to prove the the initial conditions (IC) coincides with $\rho^\pm(\cdot, 0)$. Indeed, from the estimates of $\rho^{\pm,\varepsilon,per}$ and $\dfrac{\partial \rho^{\pm,\varepsilon,per}}{\partial t}$ done in Lemmata 7.10 and 7.13, we see that $\rho^{\pm,\varepsilon}$ is ε-uniformly bounded in

$$W^{1,2}((0,T); W^{-2,2}(\mathbb{T}^2)) \hookrightarrow C^{\frac{1}{2}}([0,T); W^{-2,2}(\mathbb{T}^2)),$$

where $W^{-2,2}(\mathbb{T}^2)$ is the dual of $W^{2,2}(\mathbb{T}^2)$. It follows that, there exists a constant C independent on ε, such that, for all $t, s \in [0,T)$:

$$\|\rho^{\pm,\varepsilon,per}(t) - \rho^{\pm,\varepsilon,per}(s)\|_{W^{-2,2}(\mathbb{T}^2)} \leq C|t-s|^{\frac{1}{2}}.$$

In particular if we set $s = 0$, we have

$$\|\rho^{\pm,\varepsilon,per}(t) - \rho_0^{\pm,\varepsilon,per}\|_{W^{-2,2}(\mathbb{T}^2)} \leq Ct^{\frac{1}{2}}. \tag{9.32}$$

Now we pass to the limit in (9.32). Indeed, the functions $\rho^{\pm,\varepsilon,per}$ and $\rho_0^{\pm,\varepsilon,per}$ are ε-uniformly bounded in $W^{1,2}((0,T);W^{-2,2}(\mathbb{T}^2))$ and $W^{-2,2}(\mathbb{T}^2)$ respectively. Moreover we know that $\rho^{\pm,\varepsilon,per}-\rho_0^{\pm,\varepsilon,per}$ converges weakly in $L^2(\mathbb{T}^2\times(0,T))$ to $(\rho^{\pm,per}-\rho_0^{\pm,per})$.

Therefore, we can extract a subsequence still denoted by $(\rho^{\pm,\varepsilon,per}-\rho_0^{\pm,\varepsilon,per})$, that weakly converges in $W^{1,2}((0,T);W^{-2,2}(\mathbb{T}^2))$ to $(\rho^{\pm,per}-\rho_0^{\pm,per})$. This is possible because $W^{-2,2}(\mathbb{T}^2)=(W^{2,2}(\mathbb{T}^2))'$ and $W^{1,2}(\mathbb{T}^2)=(W^{-1,2}(\mathbb{T}^2))'$. In particular this subsequence converges, for all $t\in(0,T)$, weakly * in $L^\infty((0,t);W^{-2,2}(\mathbb{T}^2))$, and consequently it verifies (see for instance H. Brezis [21, Prop. 3.12]),

$$\|\rho^{\pm,per}-\rho_0^{\pm,per}\|_{L^\infty((0,t);W^{-2,2}(\mathbb{T}^2))}\leq \liminf \|\rho^{\pm,\varepsilon,per}-\rho_0^{\pm,\varepsilon,per}\|_{L^\infty((0,t);W^{-2,2}(\mathbb{T}^2))}\leq Ct^{\frac{1}{2}}.$$

From (9.32) we deduce that

$$\|\rho^{\pm,per}(t)-\rho_0^{\pm,per}\|_{W^{-2,2}(\mathbb{T}^2)}\leq Ct^{\frac{1}{2}}.$$

Which proves that $\rho^{\pm}(\cdot,0)=\rho_0^{\pm}$ in $\mathcal{D}'(\mathbb{R}^2)$. □

Remark 9.4
In step **1.** *of the proof, we indirectly used the fact that $\bar\rho^\varepsilon$ is bounded in $L^2((0,T);W^{1,2}(\mathbb{T}^2))$ and $\dfrac{\partial\bar\rho^\varepsilon}{\partial t}$ is bounded in $L^2((0,T);W^{-1,2}(\mathbb{T}^2))$. The usual compactness result (see L. C. Evans [51, P. 5.9.2]) asserts that we have compactness of the sequance in $C((0,T);L^2(\mathbb{T}^2))$. Here we work in dimension 2, and we use another result which asserts that we have, in particular, compactness in $L^1((0,T);EXP_\beta(\mathbb{T}^2))$ for every $1<\beta<2$.*

Remark 9.5 (BMO **times** $\mathcal{H}^1$)
We notice, using Lemma 7.5 and Remark 7.9, that we can also define the bilinear term of our system as the product duality between $L^2((0,T);\mathcal{H}^1(\mathbb{T}^2))$ and $L^2((0,T);BMO(\mathbb{T}^2))$.

Remark 9.6
In our proof, we have indirectly used a kind of compensated compactness technic for Hardy spaces. This technic allows to pass to the limit in a scalar product $B.E$ "weak times weak", if we have some regularity conditions on "div E" and on "curl B" (see R. Coifman et al. [32]). In our case, we do not have enough regularity to do so.

10 Appendix

Tis section is devoted to the proof of a generalised Hölder inequality, of which Lemma 4.3 is a particular case. The proofs of Theorem 10.1 and Lemma 10.2 are an amalgam of argument found in R. O'Neil [114, Th 2.3], M. M. Rao, Z.D. Ren [120, Th 7, Page 64] and J. Hogan et al. [76, Th A.1].

Theorem 10.1
Suppose A, B and C be three Young's functions (see Sub-section 4.1) for which there exist positive contants c and d such that,

i) $B^{-1}(t)C^{-1}(t) \leq A^{-1}(t)$ *for all* $t > 0$, *and*

ii) $A(\frac{t}{d}) \leq \frac{1}{2}A(t)$ *for all* $t > 0$.

Moreover, if $f \in L_B(\mathbb{T}^2)$ *and* $g \in L_C(\mathbb{T}^2)$, *then* $fg \in L_A(\mathbb{T}^2)$ *and*

$$\|fg\|_{L_A(\mathbb{T}^2)} \leq \|f\|_{L_B(\mathbb{T}^2)}\|g\|_{L_C(\mathbb{T}^2)}.$$

As a preliminary to the proof of the theorem, we have the follxing lemma :

***Lemma* 10.2**
Let A, B and C be as above. Then, for all $s, t > 0$,

$$A\left(\frac{st}{c}\right) \leq B(s) + C(t).$$

Proof of Lemma 10.2 :

Let $u = B(s)$ and $v = C(t)$. Then

$$st = B^{-1}(s)C^{-1}(t) \leq B^{-1}(u+v)C^{-1}(u+v) \leq cA^{-1}(u+v).$$

Dividing by c and applying A to both sides gives the result. □

Proof of Theorem 10.1 :

Note that if $f \in L_A(\mathbb{T}^2)$, the monotonicity of A and an application of the monotone convergence theorem gives us that $\displaystyle\int_{\mathbb{T}^2} A\left(\frac{|f(x)|}{\|f\|_{L_A(\mathbb{T}^2)}}\right) \leq 1$. Hence, from the definition of the Luxemburg norm (see Sub-section 4.1)

$$\int_{\mathbb{T}^2} A\left(\frac{|f(x)g(x)|}{c\|f\|_{L_B(\mathbb{T}^2)}\|g\|_{L_C(\mathbb{T}^2)}}\right) \leq \int_{\mathbb{T}^2} B\left(\frac{|f(x)|}{\|f\|_{L_B(\mathbb{T}^2)}}\right) + \int_{\mathbb{T}^2} C\left(\frac{|g(x)|}{\|g\|_{L_C(\mathbb{T}^2)}}\right) \leq 2.$$

We therefore have

$$\int_{\mathbb{T}^2} A\left(\frac{|f(x)g(x)|}{cd\|f\|_{L_B(\mathbb{T}^2)}\|g\|_{L_C(\mathbb{T}^2)}}\right) \leq 1,$$

and, again by the definition of the Luxemburg norm, we have the result. □

Proof of Lemma 4.3 :

To prove the generalised Hölder inequality of Lemma 4.3, we need only show that if $B(t) = t(\log(e+t))^\beta$ then $B^{-1}(t) \approx t(\log(e+t))^{-\beta}$. To see this, simply note that if $t = s(\log(e+s))^\beta$, there exist constants $0 < c_1(\beta) \leq c_2(\beta) < \infty$ such that for all $s > 0$,

$$c_1 \log(e+t) \leq \log(e+s) \leq c_2 \log(e+t).$$

Then $t = s(\log(e+s))^\beta \approx s(\log(e+t))^\beta$ and solving for s gives $s \approx t(\log(e+t))^{-\beta}$. Now apply Theorem 10.1 with $A = t$, $B = e^{t^2} - 1$ and $C = t(\log(e+t))^{\frac{1}{2}}$ to prove (i) and we re-use that with $A = t(\log(e+t))^{\frac{1}{2}}$, $B = e^{t^2} - 1$ and $C = t\log(e+t)$ to prove (ii). This completes the proof. □

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Chapitre 6

Simulations numériques de la dynamique des densités de dislocations pour un modèle bidimensionnel

Ce chapitre est un travail en collaboration avec M. El Rhabi et P. Hoch.
Dans ce chapitre, nous présentons quelques simulations numériques du modèle de Groma-Balogh 2D.

Simulations numériques de la dynamique des densités de dislocations pour le modèle de Groma-Balogh bidimensionnel

Résumé

Dans ce chapitre, nous présentons nos premiers essais de simulation numérique du modèle de Groma-Balogh 2D. Plus précisément, nous nous intéressons au modèle (2.6) introduit dans le chapitre 5 avec des conditions aux limites périodiques. Cette simulation montre l'évolution de la densité de dislocations dans le tore $\mathbb{T}^2 = \mathbb{Z}^2/\mathbb{R}^2$, sous l'effet d'une contrainte constante.

1 Rappel du modèle

Nous rappelons maintenant le modèle de Groma-Balogh. Dans le chapitre 5, section 2 (en particulier les équations (2.6), (2.8) et (2.12)), nous avons introduit le modèle de Groma-Balogh. Ce modèle est décrit par un système de transport non-linéaire dont la vitesse est calculée à partir de l'équation de l'élasticité linéaire. Plus précisément, le modèle est régi par le système couplé suivant :

$$\begin{cases} \dfrac{\partial \rho^{\pm}}{\partial t} = \pm \sigma_{12} \dfrac{\partial \rho^{\pm}}{\partial x_1} & \text{sur } \mathbb{R}^2 \times (0,T), \\[2ex] \sigma_{12} = \mu \left(\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} \right) - (\rho^+ - \rho^-) \right) & \text{sur } \mathbb{R}^2 \times (0,T), \\[2ex] \mu \Delta u + (\lambda + \mu) \nabla(\operatorname{div} u) = \mu \begin{pmatrix} \dfrac{\partial}{\partial x_2}(\rho^+ - \rho^-) \\[2ex] \dfrac{\partial}{\partial x_1}(\rho^+ - \rho^-) \end{pmatrix} & \text{sur } \mathbb{R}^2 \times (0,T), \end{cases} \tag{1.1}$$

où les inconnues de ce système sont les scalaires $\rho^{\pm}$ et $u = (u_1, u_2)$, le déplacement à l'instant t et à la position $x = (x_1, x_2)$. Ici les dérivées de $\rho^{\pm}$ par rapport à x_1, $\dfrac{\partial \rho^{\pm}}{\partial x_1}$ représentent les densités de dislocations de type $\pm$. Les constantes μ et λ sont

les coefficients de Lamé.

Pour la résolution de ce système, on considère les conditions initiales suivantes :

$$\rho^{\pm}(x, t = 0) = \rho_0^{\pm,per}(x) + L_0 x_1, \quad u(x, t = 0) = u_0(x),$$

où $\rho_0^{\pm,per}$ est une fonction 1-pérodique en x_1 et en x_2, L_0 est une constante donnée et u_0 est un déplacement 1-périodique et à moyenne nulle.

2 La discrétisation numérique

Dans cette section, nous présentons brièvement la discrétisation du système (1.1). Ici, on choisit des pas de discrétisation en temps et en espace fixe, noté Δt, Δx respectivement. On définit alors la grille :

$$\Xi = \{(i_1\Delta x, i_2\Delta x),\ (i_1, i_2) \in \mathbb{N}^2 \text{ telles que } i_1, i_2 \leq \tfrac{1}{\Delta x}\}, \quad \Xi_T = \Xi \times \{0, ..., (\Delta t)N_T\},$$

où N_T est la partie entière de $T/\Delta t$ et on suppose pour simplifier que $\frac{1}{\Delta x}$ est entière. On note respectivement par $\rho_I^{\pm,n}$ et $u_I^n = (u_{1,I}^n, u_{2,I}^n)$ les approximations numériques des fonctions $\rho^{\pm}$ et u au point (x_I, t_n) avec $x_I = (i_1\Delta x, i_2\Delta x)$ et $t_n = n(\Delta t)$.

Pour l'approximation numérique de la partie transport du modèle, nous avons choisi le schéma itératif, basé sur un schéma de type "Upwind", où pour tout $I = (i_1, i_2)$ tel que $(i_1, i_2) \in \mathbb{N}^2$, $i_1, i_2 \leq \frac{1}{\Delta x}$ et pour tout $n \in \mathbb{N}$, $0 < n \leq N_T - 1$, nous avons :

$$\rho_I^{\pm,n+1} = \rho_I^{\pm,n} \pm \Delta t\ (\sigma_{12})_I^n \begin{cases} E^{\pm}\left(D_1^+\rho_I^{\pm,n}, D_1^-\rho_I^{\pm,n}\right) & \text{si } (\sigma_{12})_I^n \geq 0, \\ E^{\mp}\left(D_1^+\rho_I^{\pm,n}, D_1^-\rho_I^{\pm,n}\right) & \text{sinon,} \end{cases} \tag{2.2}$$

où

$$(\sigma_{12})_I^n = \mu\left(\frac{u_{1,I^{+,2}}^n - u_{1,I}^n}{\Delta x} + \frac{u_{2,I^{+,1}}^n - u_{2,I}^n}{\Delta x} - (\rho_I^{+,n} - \rho_I^{-,n})\right),$$

tel que $I^{+,2} = (i_1, i_2 + 1)$ et $I^{+,1} = (i_1 + 1, i_2)$. De plus, les u_I^n sont calculées en utilisant une décomposition en série de Fourier pour l'équation d'élasticité linéaire. En effet, cette décomposition semble naturelle pour la discrétisation de l'équation d'élasticité dès que nous considèrons le fait que la solution soit 1-périodique et à moyenne nulle (c'est-à-dire le premier coefficient de la décomposition en série de Fourier est nul).

$E^{\pm}$ sont des approximations monotones proposées par S. Osher et J. A. Sethian [116] (nous pouvons également utiliser celles proposées par E. Rouy, A. Tourin [121]) :

$$E^{+}(P,Q) = \left(\max(P,0)^2 + \min(Q,0)^2\right)^{\frac{1}{2}},$$

$$E^{-}(P,Q) = \left(\min(P,0)^2 + \max(Q,0)^2\right)^{\frac{1}{2}}.$$

Les termes $D_1^+\rho_I^{\pm,n}$, $D_1^-\rho_I^{\pm,n}$ sont des approximations appropriées des $\dfrac{\partial\rho^{\pm}}{\partial x_1}$ pris au point x_I :

$$D_1^+\rho_I^{\pm,n} = \frac{\rho_{I^{+,1}}^{\pm,n} - \rho_I^{\pm,n}}{\Delta x},$$

$$D_1^-\rho_I^{\pm,n} = \frac{\rho_I^{\pm,n} - \rho_{I^{-,1}}^{\pm,n}}{\Delta x}.$$

où $I^{-,1} = (i_1 - 1, i_2)$. Les conditions initiales naturelles du schéma sont les suivantes :

$$\rho_I^{\pm,0} = \rho_0^{\pm,per}(x_I) + L_0(i_1\Delta x), \quad u_I^0 = u_0(x_I),$$

et nous supposons aussi des conditions aux limites 1-périodiques pour les approximations $\rho_I^{\pm,n}$, $n \in \mathbb{N}$ et $0 \leq n \leq N_T - 1$.

3 Simulations numériques

Dans ce paragraphe, nous nous intéressons à l'évolution des densités de dislocations sous l'effet d'une contrainte de cisaillement uniformément appliquée.

Dans la figure suivante, nous nous plaçons dans un pavé de ce matériau où les dislocations sont concentrées par paquets (voir Figure 6.1) et nous proposons de regarder leurs évolutions au cours du temps. Nous constatons alors, comme nous aurions pu le prévoir intuitivement, que les dislocations se distribuent uniformément dans tout le pavé (c'est-à-dire que les densités de dislocations de deux types deviennent constantes en temps long voir Figure 6.3). Nous remarquons que ces simulations sont cohérentes avec les simulations numériques présentées dans chapitre 3, section 5 pour le modèle unidimensionnel. Rappelons que pour le schéma unidimensionnel du chapitre 3 nous avons établi un résultat de convergence. Par contre, pour le schéma bidimensionnel nous n'avons pas un tel résultat.

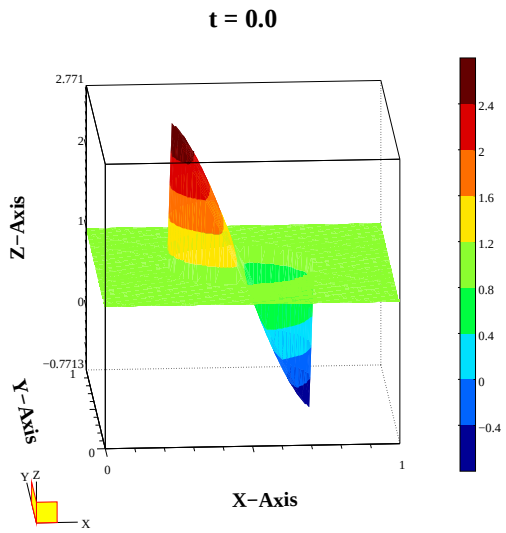

FIG. 6.1 – Les densités de dislocations de type (+) initialement.

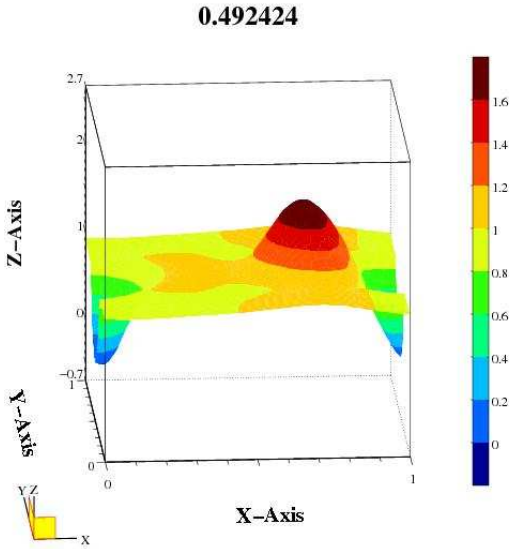

FIG. 6.2 – Les densités de dislocations de type (+) en un temps intermédiaire.

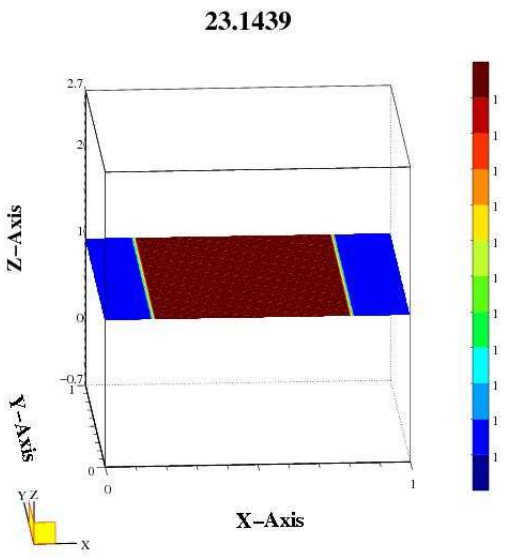

FIG. 6.3 – Les densités de dislocations de type (+) au temps final.

Remarque 3.1 *Jusqu'à présent, il nous semble qu'aucune étude publiée n'a eu pour sujet la mise en œuvre d'un schéma numérique pour ce problème.*

Conclusion et perspectives

Dans cette thèse, nous avons étudié plusieurs modèles à travers lesquels nous avons pu avoir une meilleure compréhension de la dynamique des densités de dislocations. Notre travail nous a permis d'obtenir des résultats pouvant être par la suite complétés ou approfondis. Dans ce qui va suivre, nous allons présenter quelques questions qui restent ouvertes et des directions de travail futur.

1. Système unidimensionnel :
Concernant le système (2×2), représenté dans les sous-sections 2.1 et 2.2, nous avons pu obtenir un résultat d'existence globale et d'unicité d'une solution par l'intermédiaire de deux approches : la première est celle de Sobolev et la seconde est celle de solution de viscosité. Nous avons également proposé un schéma numérique où nous avons montré une estimation d'erreur entre la solution continue et la solution discrète. Comme nous l'avons présenté dans la simulation numérique, la densité de dislocation tend vers une constante à l'infini. Cette constante est L_0 la densité totale de dislocations. Une preuve théorique d'un tel résultat sera une prochaine piste à explorer.

Concernant le système $(M \times M)$, présenté dans la sous-section 2.3, nous avons montré l'existence globale de solutions continues croissantes. L'unicité de la solution semble fortement liée à l'existence d'une solution régulière (Lipschitz). Les solutions données dans le Théorème 2.9 ne créent pas de chocs car elles sont continues. Dans cette situation, il est naturel de se poser la question de l'unicité des solutions, qui reste une question ouverte.

D'un point de vue numérique, nous sommes en train de mettre au point un schéma qui conserve l'estimation entropique de gradient (2.19), dans une approche discrète. Cela nous permettra de montrer en utilisant le Théorème 2.9 un résultat de convergence d'un tel schéma. Ce genre de schéma est également intéressant pour la discrétisation du modèle bidimensionnel, étant donné que l'existence de solutions dans le cas bidimensionnel est basée sur le même type d'estimation entropique de gradient.

2. Système bidimensionnel :
Suite à notre étude du modèle bidimensionnel, nous avons montré un résultat d'existence globale de solutions. L'unicité ou la non-unicité des solutions dans ce cadre est une question intéressante qui reste ouverte pour la dynamique des densités de dislocations. Étant donné que les solutions données par le Théorème 3.4, ne sont pas nécessairement continues, contrairement au cas unidimensionnel, cette question nous semble difficile à traiter. Toutefois, nous pouvons aboutir à un résultat d'existence et d'unicité en temps court dans certain espaces de Hölder. Ceci constituera le sujet d'un futur travail.

3. Autre étude :
Nous essayons aussi de développer une méthode numérique de type Fast Marching. Notre but est de discrétiser l'evolution d'un front transporté pat un champ de vecteur quelconque dépendant de l'espace et du temps. L'objectif est de réussir à montrer un résultat de convergence pour cette méthode, en s'inspirant du travail de E. Carlini et al. [27] développé dans le cadre de propagation d'un front avec une vitesse normale qui dépend de l'espace et du temps.

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