THE QUEUE WITH IMPATIENCE: CONSTRUCTION OF THE STATIONARY WORKLOAD UNDER FIFO

PASCAL MOYAL,* Université de Technologie de Compiègne

Abstract

In this paper, we study the stability of queues with impatient customers having a single server operating in FIFO. We first give a sufficient condition for the existence of a stationary workload in the case of impatience until the beginning of service. We then provide a weaker condition of existence on an enriched probability space using the Theory of Anantharam et al. (see [1, 2]). The case of impatience until the end of service is as well investigated.

Keywords: Stochastic recursions; Stationary solutions; Queues with impatience; Renovative events; Enrichment of probability space.

2000 Mathematics Subject Classification: Primary 60F17

Secondary 60K25, 60B12

1. Introduction

In this paper, we address the question of stability for queueing systems with impatient customers: the customers agree to wait for their service only during a limited period of time. They abandon from the system whenever their patience ends before they could reach the service booth. Such models are particularly adequate to describe operating systems under sharp delay requirements: multimedia and time sensitive telecommunication and computer networks, on-line audio/video traffic flows, call centres or supply chains.

* Postal address: Laboratoire de Mathématiques Appliquées de Compiègne
Université de Technologie de Compiègne
Département Génie Informatique
Centre de Recherches de Royallieu
BP 20 529
60 205 COMPIEGNE Cedex
FRANCE
We construct explicitly a stationary state for these systems in the particular case of a single server obeying the FIFO (First In, First Out) discipline. To that end, we study a stochastically recursive sequence representing the workload seen by an arriving customer. This sequence and its dynamics have been thoroughly studied in the GI/GI/1 case in [5] and [3]. In the G/G/1 context, the workload sequence is driven by a non-monotonic recursive equation (eq. (2.11) hereafter), hence a construction of Loynes’s type, using a backwards recurrence scheme, is not possible. We then use more sophisticated techniques to construct a stationary workload: (i) Borovkov’s theory of renovating events (see [7], [8], [4]) provides a sufficient condition for the existence and uniqueness of a finite stationary workload. Under this condition, we can thus construct a stationary loss probability $\pi$, and provide bounds for $\pi$ (eq. (2.18)). (ii) We provide a weaker condition for the existence of a stationary workload on the enriched probability space $\Omega \times \mathbb{R}_+$ (where $\Omega$ is the Palm probability space of reference) using Anantharam and Konstantopoulos’ construction (see [1, 2]), which is based on tightness techniques.

In both cases, we use the fact that the workload sequence is strongly dominated by another one, that is driven by a monotonic recursive equation (eq. (2.2)). Then the coupling of the dominating sequence with a unique stationary state allows us to construct the stationary state of the dominated sequence.

We address as well the case, where the customers are impatient until the end of their service: they abandon from the queue whenever their patience ends before a server could complete their service. This case turns out to be more simple, in that the workload sequence is driven by a monotonic and continuous recursive random mapping. Then the stability problem can be handled by Loynes’ scheme.

This paper is organized as follows. In section 2 we make precise our basic assumptions, and introduce the queue with impatience until the beginning of service. In 2.2, we address the problem of existence of construction points. We provide a sufficient condition for existence and uniqueness of a stationary workload in 2.4, and for the existence on an enriched probability space in 2.6. In section 3 we study the case of impatience until the end of service.
2. The queue with impatient customers

Let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}^*$ denote the sets of real numbers, integers, non-negative integers and positive integers, respectively. We consider a queue with impatient customers $G/G/1/1+G(b)$ (according to Barrer’s notation, see [6]). On the probability space $(\Omega, \mathcal{F}, P)$, furnished with the measurable bijective flow $(\theta_t)_{t \in \mathbb{R}}$ under which $P$ is stationary and ergodic, consider the $\theta_t$-compatible point process $N$, whose points $\{T_n\}_{n \in \mathbb{Z}}$ represent the arrivals of the customers $\{C_n\}_{n \in \mathbb{Z}}$, with the convention that $T_0$ is the last arrival before time $t = 0$. The inter-arrivals are denoted by $\xi_n = T_{n+1} - T_n$, for all $n \in \mathbb{Z}$. The process $N$ is marked by the sequence $\{\sigma_n\}_{n \in \mathbb{Z}}$ of non-negative r.v.’s representing the service durations requested by the customers. The queueing system has a single non-idling server, and its buffer is of infinite capacity. The customers are impatient until the beginning of their service, in that they leave the system if they do not reach the service booth before a given deadline. In other words, customer $C_n$ agrees to wait in line for a given period of time, say $D_n$ (his initial patience) and if the server is not available during this period, he leaves the system forever at time $T_n + D_n$. $\{D_n\}_{n \in \mathbb{Z}}$ is a sequence of non-negative marks of $(N_t)_{t \in \mathbb{R}}$. It is furthermore assumed throughout that $\xi_0$, $\sigma_0$ and $D_0$ are integrable and that $P[\xi_0 > 0] > 0$. We denote by $X_t$ the number of customers in the system (or congestion) at $t$, for all $t \in \mathbb{R}$. We assume throughout that a customer can not be eliminated anymore as soon as he enters the service booth, even if his deadline is reached during his service.

Let $(\Omega, \mathcal{F}, P_0, \theta)$ be the Palm space of $N(\sigma, D)$, where $\theta := \theta_{T_1}$ is the associated bijective discrete flow (denote $\theta^{-1}$, its measurable inverse). Then, $P_0$ is stationary and ergodic under $\theta$, i.e. for all $A \in \mathcal{F}$, $P_0[\theta^{-1}A] = P_0[A]$ and all $A$ that are $\theta$-invariant (i.e. such that $\theta A = A$) are of probability 0 or 1. Note that according to these axioms, all $\theta$-contracting events (such that $P_0[A^c \cap \theta^{-1}A] = 0$) are of probability 0 or 1. We denote $\theta^n = \theta \circ \theta \circ \ldots \circ \theta$ and $\theta^{-n} = \theta^{-1} \circ \theta^{-1} \circ \ldots \circ \theta^{-1}$ for all $n \in \mathbb{N}$. Note, that the sequence $\{\xi_n, \sigma_n, D_n\}_{n \in \mathbb{Z}}$ is stationary in that for all $n$, $\xi_n = \xi \circ \theta^n$, $\sigma_n = \sigma \circ \theta^n$ and $D_n = D \circ \theta^n$, where $\xi := \xi_0$, $\sigma := \sigma_0$ and $D := D_0$. Note moreover that the r.v.’s $\xi$, $\sigma$ and $D$ are
\( \mathbf{P}^0 \)-integrable, and that \( \mathbf{P}^0 [\xi > 0] > 0 \). We say that two sequences of r.v.’s \( \{X_n\}_{n \in \mathbb{N}} \) and \( \{Y_n\}_{n \in \mathbb{N}} \) couple when there exists a \( \mathbf{P}^0 \)-a.s. finite index \( N \) such that they coincide for all \( n \geq N \). We say that there is strong backwards coupling between \( \{X_n\}_{n \in \mathbb{N}} \) and \( \{Y \circ \theta^n\}_{n \in \mathbb{N}} \) provided that for some \( \mathbf{P}^0 \)-a.s. finite \( \tau \), \( X_n \circ \theta^{-n} = Y \) for any \( n \geq \tau \).

As well known (see e.g. [4]), the problems of existence and uniqueness of a stationary regime for random processes observed at the arrival times can be formulated in a simple manner on \( (\Omega, \mathcal{F}, \mathbf{P}^0, \theta) \). Let \( Y \) be a \( \mathbb{R} \)-valued r.v., and \( \phi \) be a random mapping \( \mathbb{R} \mapsto \mathbb{R} \), for which we shall emphasize the randomness, when needed, by denoting \( \phi[\omega](x) \), the image of \( x \) through \( \phi \) for the sample \( \omega \). The stochastic recursive sequence (SRS) initiated by \( Y \) and driven by \( \phi \) is defined by

\[
\begin{align*}
X^Y_0(\omega) &= Y(\omega), \quad \mathbf{P}^0 - \text{a.s.,} \\
X^Y_{n+1}(\omega) &= \phi[\theta^n \omega](X^Y_n(\omega)), \quad n \in \mathbb{N}, \quad \mathbf{P}^0 - \text{a.s.}
\end{align*}
\]

Then, the problem of existence of a stationary regime for this sequence, amounts to that of a r.v. \( Y \) such that \( X^Y_n = Y \circ \theta^n \) for all \( n \in \mathbb{N} \). This is in turn equivalent to say that \( Y \) solves the functional equation

\[
Y \circ \theta = \phi(Y), \quad \mathbf{P}^0 - \text{a.s.}
\]

Throughout, the stability study of the queue with impatient customers will be handled under these settings, for several processes of interest.

### 2.1. Preliminary Result

Let us denote \( x \lor y = \max(x, y) \), \( x \land y = \min(x, y) \) and \( x^+ = x \lor 0 \), for any \( x, y \in \mathbb{R} \). On \( (\Omega, \mathcal{F}, \mathbf{P}^0, \theta) \), let \( \alpha \) and \( \beta \) be two integrable \( \mathbb{R}_+ \)-valued r.v.’s, such that \( \mathbf{P}^0 [\beta > 0] > 0 \). Let \( F_{\alpha, \beta} \) be the real-valued random mapping defined for \( x \in \mathbb{R} \) by

\[
F_{\alpha, \beta}(x) = [x \lor \alpha - \beta]^+.
\]  

(2.1)

The SRS \( \{Y^Z_n\}_{n \in \mathbb{N}} \) initiated by \( Z \) and driven by \( F_{\alpha, \beta} \) is stationary if and only if \( Z \) solves the equation

\[
Z \circ \theta = F_{\alpha, \beta}(Z).
\]  

(2.2)

We have the following result (which completes Lemma 5 in [11]).
Lemma 1. There exists a unique $P^0$-a.s. finite solution $Y_{\alpha,\beta}$ of (2.2), given by

$$Y_{\alpha,\beta} := \left[ \sup_{j \in \mathbb{N}} \left( \alpha \circ \theta^j - \sum_{i=1}^{j} \beta \circ \theta^{-i} \right) \right]^+.$$  \hspace{1cm} (2.3)

Moreover, for any $P^0$-a.s. finite and nonnegative r.v. $Z$, the sequence $\{Y_n^Z\}_{n \in \mathbb{N}}$ couples in the strong backwards sense with $\{Y_{\alpha,\beta} \circ \theta^n\}_{n \in \mathbb{N}}$, and there exists $P^0$-a.s. an infinity of indices such that $Y_n^Z = 0$ if and only if

$$P^0 [Y_{\alpha,\beta} = 0] > 0.$$  \hspace{1cm} (2.4)

Proof. Equation (2.2) can be handled by Loynes’s construction (see [10], [4]) since the mapping $F_{\alpha,\beta}$ is $P^0$-a.s. continuous and non-decreasing. Hence $Y_{\alpha,\beta}$ classically reads as the $P^0$-a.s. limit of Loynes’s sequence, defined by $\{Y_0 \circ \theta^{-n}\}_{n \in \mathbb{N}}$.

It is routine to check from Birkhoff’s ergodic theorem (and the fact that $\beta$ is not identically zero) that $Y_{\alpha,\beta}$ is $P^0$-a.s. finite. Finally, the coupling property follows from the fact that for all non-negative r.v.’s $Z$ that are $P^0$-a.s finite (and in particular, for $Z = Y_{\alpha,\beta}$),

$$\{Y_n^Z \neq Y_n^0 \text{ for all } n \in \mathbb{N}\} = \left\{ Y_n^Z = Z - \sum_{i=0}^{n-1} \beta \circ \theta^i > 0 \text{ for all } n \in \mathbb{N} \right\},$$

which is of probability 0 from Birkhoff’s theorem. The last statement is a classical consequence of this coupling property under ergodic assumptions.

2.2. General case: construction points

According to the assumptions made above, the total sojourn time of customer $C_n$ does not exceed $D_n + \sigma_n$, i.e. the sum of his initial patience and the time necessary for his service. On the other hand, it is at least equal to $\sigma_n \land D_n$, i.e. the time needed for him to be lost, or immediately served. Hence, provided that $C_n$ entered the system before $t$ ($T_n \leq t$) and even though he already entered service, or left the system before $t$, his remaining maximal sojourn time at $t$ (i.e. the remaining time before his originally latest possible departure time, if not already reached) is given by $[\sigma_n + D_n - (t - T_n)]^+$, whereas his remaining minimal sojourn time at $t$ (i.e. the remaining time before his originally earliest possible departure time, if not already reached) is given by $[\sigma_n \land D_n - (t - T_n)]^+$. Hence the largest remaining maximal sojourn time (LRMST for short) at $t$ among all the customers entered before $t$ is given
by
\[ \mathcal{L}_t := \max_{n=1}^{N_t} [\sigma_n + D_n - (t - T_n)]^+ \]
and the largest remaining minimal sojourn time (LRmST for short) at \( t \), by
\[ \mathcal{M}_t := \max_{n=1}^{N_t} [\sigma_n \wedge D_n - (t - T_n)]^+. \]

The two processes \((\mathcal{L}_t)_{t \in \mathbb{R}}\) and \((\mathcal{M}_t)_{t \in \mathbb{R}}\) clearly have càdlàg paths. We define for all finite nonnegative r.v.'s \( Y \) and \( Z \) and all \( n \in \mathbb{N} \), \( L^Y_n := \mathcal{L}_{T_n} \) and \( M^Z_n := \mathcal{M}_{T_n} \), the LRMST (resp. LRmST) just before the arrival of customer \( C_n \), provided that \( L^Y_0 = Y \) (resp. \( M^Z_0 = Z \)). Then, it is easily checked that \( P^0 \)-a.s. for all \( n \in \mathbb{N} \)
\[ L^Z_{n+1} = [L^Z_n \lor (\sigma_n + D_n) - \xi_n]^+ = F_{\sigma_n + D_n, \xi_n}(L_n), \quad \text{(2.5)} \]
\[ M^Z_{n+1} = [M^Z_n \lor (\sigma_n \wedge D_n) - \xi_n]^+ = F_{\sigma_n \wedge D_n, \xi_n}(M_n), \]
using the notation introduced in (2.1). Consequently, a stationary LRMST \( L \) and a stationary LRmST \( M \) satisfy
\[ L \circ \theta = F_{\sigma + D, \xi}(L), \]
\[ M \circ \theta = F_{\sigma \wedge D, \xi}(M). \]

Consequently, in view of Lemma 1, a unique couple \((L, M)\) exists, given by
\[ L = Y_{\sigma + D, \xi} = \sup_{j \in \mathbb{N}^*} \left( \sigma_{-j} + D_{-j} - \sum_{i=1}^{j} \xi_{-i} \right)^+, \quad \text{(2.6)} \]
\[ M = Y_{\sigma \wedge D, \xi} = \sup_{j \in \mathbb{N}^*} \left( \sigma_{-j} \wedge D_{-j} - \sum_{i=1}^{j} \xi_{-i} \right)^+. \quad \text{(2.7)} \]

In particular, for any initial conditions \( Y \) and \( Z \), \( \{L^Y_n\}_{n \in \mathbb{N}} \) and \( \{M^Z_n\}_{n \in \mathbb{N}} \) couple respectively with \( \{Y_{\sigma + D, \xi} \circ \theta^n\}_{n \in \mathbb{N}} \) and \( \{Y_{\sigma \wedge D, \xi} \circ \theta^n\}_{n \in \mathbb{N}} \). Therefore, there are \( P \)-a.s. an infinity of indices such that \( L^Y_n = 0 \) if and only if \( P^0 [L = 0] > 0 \), and an infinity of indices such that \( M^Z_n = 0 \) if and only if \( P^0 [M = 0] > 0 \). Remarking now that for all initial conditions and all \( t \geq 0 \),
\[ \{\mathcal{L}_t = 0\} \subseteq \{\lambda_t = 0\} \subseteq \{\mathcal{M}_t = 0\}, \]
we have proven the following elementary result.
Theorem 1. The $G/G/s/s+G(b)$ queue empties $\mathbf{P}^0$-a.s. an infinite number of times if
\[ \mathbf{P}^0 \left[ \sup_{j \in \mathbb{N}^*} \left( \sigma_j - D_j - \sum_{i=1}^{j} \xi_i \right) \leq 0 \right] > 0, \]  
and only if
\[ \mathbf{P}^0 \left[ \sup_{j \in \mathbb{N}^*} \left( \sigma_j \wedge D_j - \sum_{i=1}^{j} \xi_i \right) \leq 0 \right] > 0. \]

2.3. The single-server FIFO queue: Workload sequence

Until the end of section 2, we assume that the single server obeys the FIFO (First in, first out) discipline. For all $t \in \mathbb{R}$, denote $\mathcal{W}_t$ the workload submitted to the server at time $t$, i.e. the quantity of work he still has to achieve at this time, in time unit. The process $(\mathcal{W}_t)_{t \in \mathbb{R}}$ has càdlàg paths, and we define for all $n$, $W_n = \mathcal{W}_{T_n}$. Its value at $t$ equals the work brought by the customers arrived up to $t$, and who will eventually be served, since the other ones won’t ever reach the server. Under the FIFO discipline, the served customers are those who find a workload less than their patience upon arrival. In-between arrival times, the process $(\mathcal{W}_t)_{t \in \mathbb{R}}$ decreases at unit rate. Hence, the workload sequence is driven by the recursive equation
\[ W_{n+1} = \left[ W_n + \sigma_n \mathbb{1}_{\{W_n \leq D_n\}} - \xi_n \right]^+. \]

In other words, the Workload sequence is stochastically recursive, driven by the random map $\varphi$ defined for all $x \in \mathbb{R}$ by
\[ \varphi(x) = \left[ x + \sigma \mathbb{1}_{\{x \leq D\}} - \xi \right]^+. \]

A stationary workload $W$ hence solves
\[ W \circ \theta = \varphi(W). \]

The random map $\varphi$ is not monotonic in the state variable, hence a construction of Loynes’s type is fruitless. In section 2.4, we use renovating events to provide a sufficient condition for the existence and uniqueness of a solution to (2.11). In section 2.6, we show that under weaker assumptions, a solution exists on an enlarged probability space.
2.4. Sufficient condition

**Theorem 2.** If (2.8) holds, (2.11) admits a unique finite solution \( W \), that is such that

\[
Y_{\sigma \land D, \xi} \leq W \leq Y_{\sigma + D, \xi}, \quad P^0 - \text{a.s.,}
\]

where \( Y_{\sigma + D, \xi} \) and \( Y_{\sigma \land D, \xi} \) are defined by (2.6) and (2.7), respectively. Moreover, for any initial condition \( Z \) such that \( Z \leq Y_{\sigma + D, \xi} \), \( P^0 \)-a.s., there is strong backwards coupling for \( \{W_n^Z\}_{n \in \mathbb{N}} \) with \( \{W \circ \theta^n\}_{n \in \mathbb{N}} \).

This result can be related to the classical stability result for the GI/GI/1/1+GI(b)-FIFO queue (i.e. all sequences are i.i.d. and independent of one another), namely Lemma 2, p.162 of [5] (see as well [3]). In that case, the recursive sequence \( \{W_n\}_{n \in \mathbb{N}} \) is a Markov Chain, that is proven to be ergodic whenever the generic patience \( D \) is a.s. finite (an assumption that is made throughout this paper), and whenever

\[
P[\sigma < \xi] > 0. \tag{2.12}
\]

The latter result follows from typical properties of Markov Chains (Harris Recurrence and irreducibility), and it is not surprising that our result, under more general assumptions, is weaker. More precisely, the existence of a stationary workload and the recurrence of 0 (Theorem 1) are entailed by Lemma 2 in [5] for a GI/GI/1/1+GI(b)-FIFO queue whenever (2.8) holds, and whenever either one of the r.v.’s \( \xi \) and \( \sigma \) is absolutely continuous, or \( P^0[D > 0] = 1 \). Indeed, it then follows that on an event \( A \) such that \( P^0[A] > 0 \),

\[
\sigma_{-1} - \xi_{-1} < \sigma_{-1} + D_{-1} - \xi_{-1} \leq 0,
\]

and (2.12) holds in view of the stationarity of \( \theta \).

**Proof of Theorem 2. Existence.** Let us first remark that for any \( x \in \mathbb{R}^+ \),

\[
\varphi(x) = [x + \sigma \mathbb{1}_{\{x \leq D\}} + x \mathbb{1}_{\{D < x \leq D + \sigma\}} + x \mathbb{1}_{\{D + \sigma < x\}} - \xi]^+ \\
\leq [(D + \sigma) \mathbb{1}_{\{x \leq D\}} + (D + \sigma) \mathbb{1}_{\{D < x \leq D + \sigma\}} + x \mathbb{1}_{\{D + \sigma < x\}} - \xi]^+ \tag{2.13} \\
= [x \vee (\sigma + D) - \xi]^+ = F_{\sigma + D, \xi}(x), \quad P^0\text{-a.s.}
\]

In view of the a.s. increasingness of \( F_{\sigma + D, \xi} \), we have in particular that for all \( x \leq y \),

\[
\varphi(x) \leq F_{D + \sigma, \xi}(y), \quad P^0 - \text{a.s.} \tag{2.14}
\]
and a straightforward induction shows that \( Z \leq Y_{\sigma + D, \xi} \) implies

\[
W_n^Z \leq Y_{\sigma + D, \xi} \circ \theta^n, \ n \geq 0.
\]

Therefore, denoting for all \( n \) the event

\[
\mathcal{A}_n := \{ Y_{\sigma + D, \xi} \circ \theta^n = 0 \},
\]

the sequence \( \{ \mathcal{A}_n \}_{n \in \mathbb{N}} \) is a sequence of renovating events of length 1 for \( \{ W_n^Z \}_{n \in \mathbb{N}} \) (see [4], p.115, [7], [8]), since

\[
\mathcal{A}_n \subseteq \{ W_n^Z = 0 \}, \ n \geq 0.
\]

Moreover, this sequence is stationary in the sense that for all \( n \geq 0, \)

\[
\mathcal{A}_n = \theta^{-n} \mathcal{A}_0, \ n \geq 0.
\]

Consequently, since (2.8) amounts to \( \mathbf{P}^0 [\mathcal{A}_0] > 0 \), this is from [4], Theorem 2.5.3., a sufficient condition for the existence of a solution \( W \) to (2.11), and for strong backwards coupling to occur for \( \{ W_n^Z \}_{n \in \mathbb{N}} \) with \( W \).

**Uniqueness.** Let \( W \) be a solution of (2.11). First, assuming that \( W > D \), \( \mathbf{P}^0 \)-a.s.

(which implies in particular that \( W \circ \theta > 0, \mathbf{P}^0 \)-a.s.) yields to

\[
W \circ \theta = W - \xi, \ \mathbf{P}^0 - \text{a.s.},
\]

an absurdity in view of the Ergodic Lemma ([4], Lemma 2.2.1). Therefore, we have that

\[
\mathbf{P}^0 [W \leq D] > 0. \tag{2.15}
\]

On the other hand, in view of (2.14), on the event \( \{ W \leq Y_{\sigma + D, \xi} \}, \)

\[
W \circ \theta = \varphi(W) \leq F_{\sigma + D, \xi} (Y_{\sigma + D, \xi}) = Y_{\sigma + D, \xi} \circ \theta.
\]

Thus \( \{ W \leq Y_{\sigma + D, \xi} \} \) is \( \theta \)-contracting, whereas on \( \{ W \leq D \}, \)

\[
W \circ \theta = [(W + \sigma) \mathbb{1}_{\{ W \leq D \}} - \xi]^+ \leq [(D + \sigma) \mathbb{1}_{\{ W \leq D \}} - \xi]^+ \leq Y_{\sigma + D, \xi} \circ \theta,
\]

which shows in view of (2.15) that

\[
0 < \mathbf{P}^0 [W \leq D] \leq \mathbf{P}^0 [W \circ \theta \leq Y_{\sigma + D, \xi} \circ \theta] = \mathbf{P}^0 [W \leq Y_{\sigma + D, \xi}].
\]
Therefore, the event \( \{ W \leq Y_{\sigma+D, \xi} \} \) is \( \mathbb{P}^0 \)-almost sure. As a consequence, \( \{ W_n^W \}_{n \in \mathbb{N}} = \{ W \circ \theta^n \}_{n \in \mathbb{N}} \) admits \( \{ A_n \}_{n \in \mathbb{N}} \) as a stationary sequence of renovating events of length 1. From [4], Remark 2.5.3, \( \mathbb{P}^0 [A_0] > 0 \) implies the uniqueness property.

Finally, for any \( x \in \mathbb{R}_+ \), we have \( \mathbb{P}^0 \)-a.s. that

\[
F_{\sigma \wedge D, \xi}(x) = \left[ (\sigma \wedge D) \mathbb{I}_{\{x \leq D \wedge \sigma\}} + x \mathbb{I}_{\{x > D \wedge \sigma\}} \right] + x \mathbb{I}_{\{x \leq D\}} - \xi
\]

\[
\leq \left[ (x + \sigma) \mathbb{I}_{\{x \leq D \wedge \sigma\}} \mathbb{I}_{\{x \leq D\}} + (x + \sigma) \mathbb{I}_{\{x > D \wedge \sigma\}} \right] + x \mathbb{I}_{\{x \leq D\}} - \xi
\]

\[
= \varphi(x).
\]

(2.16)

In view of the a.s. increasingness of \( F_{\sigma \wedge D, \xi} \), this implies that on the event \( \{ Y_{\sigma \wedge D, \xi} \leq W \} \),

\[
Y_{\sigma \wedge D, \xi} \circ \theta = F_{\sigma \wedge D, \xi} (Y_{\sigma \wedge D, \xi}) \leq \varphi(W) = W \circ \theta,
\]

thus \( \{ Y_{\sigma \wedge D, \xi} \leq W \} \) is \( \theta \)-contracting. It is \( \mathbb{P}^0 \)-almost sure since it includes the event \( \{ Y_{\sigma+D, \xi} = 0 \} \) (in view of the immediate fact that \( Y_{\sigma \wedge D, \xi} \leq Y_{\sigma+D, \xi} \), \( \mathbb{P}^0 \)-a.s.).

It is customary, that under the FIFO discipline, the construction of the stationary versions of several quantities of interest can be derived from that of the workload sequence. In particular, provided that (2.8) holds, one can construct a congestion process and a departure process that are jointly compatible with the arrival process \( (N_t)_{t \in \mathbb{R}} \). Let us remark, that under condition (2.8) there exists also a stationary loss probability, denoted \( \pi(b) \), which is the probability that the waiting time proposed to a customer exceeds his initial patience, at equilibrium. This reads

\[
\pi(b) = \mathbb{P}^0 [W > D].
\]

(2.17)

With Theorem 2 in hand, we have in particular that

\[
\mathbb{P}^0 [Y_{\sigma \wedge D, \xi} > D] \leq \pi(b) \leq \mathbb{P}^0 [Y_{\sigma+D, \xi} > D].
\]

(2.18)

2.5. Counter Examples

Let us now focus on cases where condition (2.8) does not hold. As we show in the following counter-examples, it is then easy to construct examples where uniqueness, and even existence are not granted when working on the original probability space. Let \( \Omega := \{ \omega_1, \omega_2 \} \). Denote by \( \mathbb{P}^0 \) the uniform probability on \( \Omega \), and define the shift \( \theta \)}
on \((\Omega, \mathbf{P}^0)\) by \(\theta(\omega_1) = \omega_2\) and \(\theta(\omega_2) = \omega_1\). Hence \(\theta\) is clearly stationary and ergodic under \(\mathbf{P}^0\). Note, that any solution \(W\) of (2.11) satisfies

\[
W(\omega_2) = \left[ W(\omega_1) + \sigma(\omega_1) \mathbf{1}_{\{W(\omega_1) \leq D(\omega_1)\}} - \xi(\omega_1) \right]^+; \tag{2.19}
\]

\[
W(\omega_1) = \left[ W(\omega_2) + \sigma(\omega_2) \mathbf{1}_{\{W(\omega_2) \leq D(\omega_2)\}} - \xi(\omega_2) \right]^+. \tag{2.20}
\]

First Example: non-existence Let the r.v.'s \(\xi,\sigma\) and \(D\) be defined on \((\Omega, \mathbf{P}^0, \theta)\), and satisfy

\[
\begin{align*}
\xi(\omega_1) &= \xi(\omega_2) = 1; \\
\sigma(\omega_1) &> 2, \sigma(\omega_2) \neq 2; \\
D(\omega_1) &\geq 3, D(\omega_2) \leq 2.
\end{align*}
\]

First, if \(W(\omega_1) \leq 3\), then in view of (2.19),

\[
W(\omega_2) = \left[ W(\omega_1) + \sigma(\omega_1) - 1 \right]^+ = W(\omega_1) + \sigma(\omega_1) - 1.
\]

Then with (2.20),

\[
W(\omega_1) = \left[ W(\omega_1) + \sigma(\omega_1) - 2 + \sigma(\omega_2) \mathbf{1}_{\{W(\omega_2) \leq D(\omega_2)\}} \right]^+
\]

\[
= W(\omega_1) + \sigma(\omega_1) - 2 + \sigma(\omega_2) \mathbf{1}_{\{W(\omega_2) \leq D(\omega_2)\}} > W(\omega_1),
\]

which is absurd. Hence, \(W(\omega_1) > 3\), but then \(W(\omega_2) = [W(\omega_1) - 1]^+ = W(\omega_1) - 1 > 2 \geq D(\omega_2)\), which with (2.20) yields to \(W(\omega_1) = [W(\omega_1) - 2]^+ = W(\omega_1) - 2\), another absurdity. There is no solution to (2.11).

Second example: non-uniqueness Assume now that

\[
\begin{align*}
\xi(\omega_1) &= \xi(\omega_2) = 1; \\
\sigma(\omega_1) &\in [0, 2], \sigma(\omega_2) = 2 - \sigma(\omega_1); \\
D(\omega_1) &> (1 - \sigma(\omega_1))^+, D(\omega_2) \leq D(\omega_1) + \sigma(\omega_1) - 1.
\end{align*}
\]

Then, it is easily seen that for any \(x \in ((1 - \sigma(\omega_1))^+, D(\omega_1))\), the r.v. defined by

\[
W(\omega_1) = x; W(\omega_2) = x + \sigma(\omega_1) - 1
\]

is a solution to (2.11).
2.6. Weak stationarity

In this section, we show how the techniques developed in [1] allow to construct a stationary workload for the queue under weaker assumptions, on a probability space that is enriched with respect to the original one. This is done using again the stochastic comparison with the LRMST sequence (see (2.13)).

Throughout this section, suppose without loss of generality that $\Omega$ is Polish (see [1] for a precise definition). We work on the enlarged probability space $\Omega \times \mathbb{R}$, on which we define the shift

$$\tilde{\theta}(\omega, x) = (\theta \omega, \varphi[\omega](x)).$$

We say that a subset $I \subset \mathbb{R}$ is locally finite if $I \cap A$ has a finite cardinal for any interval $A$. The following result holds.

**Theorem 3.** Suppose that either (2.8), or the following conditions hold:

The r.v.'s $\sigma$ and $\xi$ are valued in a common locally finite space $I$ that includes 0, and is closed under addition.

Then, the stochastic recursion (2.11) admits a weak solution in the sense of [1], that is, a $\tilde{\theta}$-invariant probability $\tilde{P}_0$ on $\Omega \times \mathbb{R}$ whose $\Omega$-marginal is $P_0$. Therefore, on $(\Omega \times \mathbb{R})$ there exists a $\mathbb{R} \times M(\mathbb{R})$-valued r.v. $(\tilde{W}, \tilde{\varphi})$ satisfying

$$\tilde{W} \circ \tilde{\theta} = \tilde{\varphi}(\tilde{W}), \tilde{P}_0 \text{-a.s.}$$

In particular, $\left\{ \left( \tilde{W}, \tilde{\varphi} \right) \circ \tilde{\theta}^n \right\}_{n \in \mathbb{N}}$ is stationary under $\tilde{P}_0$, and the $\Omega$-marginal of $\left\{ \tilde{\varphi}[\tilde{\theta}^n.] \right\}_{n \in \mathbb{N}}$ is the distribution of $\left\{ \varphi[\theta^n.] \right\}_{n \in \mathbb{N}}$.

**Proof.** We aim to apply Theorem 1 of [1], whose corrected version is presented in [2]. Let us check that its hypotheses are met in our case.

First, the sequence $\{L^n_0\}_{n \in \mathbb{N}}$ is tight since it converges weakly. On the other hand, (2.14) implies using an immediate induction that $P_0$-a.s. $W^n_0 \leq L^n_0$ for all $n \in \mathbb{N}$. Hence, $\{W^n_0\}_{n \in \mathbb{N}}$ is tight, since for all $\varepsilon > 0$, there exists $M_\varepsilon$ such that for all $n \in \mathbb{N}$,

$$P_0 [W^n_0 \leq M_\varepsilon] \geq P_0 [L^n_0 \leq M_\varepsilon] \geq 1 - \varepsilon.$$

Define now on $\Omega \times \mathbb{R}$ the random variables

$$\tilde{W}(\omega, x) := x, \quad \tilde{\varphi}[\omega, x] := \varphi[\omega],$$
and for all \( n \in \mathbb{N} \),
\[
\tilde{W}_n(\omega, x) := \tilde{W} \left( \hat{\theta}^n(\omega, x) \right).
\]

Remark that for all \( n \in \mathbb{N} \), \( A \in \mathcal{F} \) and \( B \in \mathcal{B}(\mathbb{R}) \),
\[
P^0 \otimes \delta_0 \left[ \hat{\theta}^{-n} (A \times \mathbb{R}) \right] = P^0 [\hat{\theta}^{-n} A] = P^0 [A]
\]
and
\[
P^0 \otimes \delta_0 \left[ \hat{\theta}^{-n} (\Omega \times B) \right] = P^0 \otimes \delta_0 \left[ \{ (\omega, x) \in \Omega \times \mathbb{R}; W^x_n(\omega) \in B \} \right] = P^0 \left[ W^0_n \in B \right].
\]

Hence, the probability distributions \( \{ (P^0 \otimes \delta_0) \circ \hat{\theta}^{-n} \}_{n \in \mathbb{N}} \) on \( \Omega \times \mathbb{R} \) have \( \Omega \)-marginal \( P^0 \) and \( \mathbb{R} \)-marginals, the distributions of \( \{ W^0_n \}_{n \in \mathbb{N}} \), which form a tight sequence. The sequence \( \{ (P^0 \otimes \delta_0) \circ \hat{\theta}^{-n} \}_{n \in \mathbb{N}} \) is thus tight. Note, that this entails in particular the tightness of the sequence
\[
\{ \tilde{Q}_n \}_{n \in \mathbb{N}} := \left\{ \frac{1}{n} \sum_{i=0}^{n-1} (P^0 \otimes \delta_0) \circ \hat{\theta}^{-i} \right\}_{n \in \mathbb{N}}
\]

We now aim to check condition (A3) p.272 of [2]. In that purpose, let us define for all \( p \in \mathbb{N}^* \),

(i) \( V_p = \{ (\omega, x) \in \Omega \times \mathbb{R}; D(\omega) < x < D(\omega) + 2^{-p} \} \),

(ii) for any \( (\omega, x) \in \Omega \times \mathbb{R} \),
\[
f_p(\omega, x) = \text{1}_{\{x \leq D(\omega)\}} + \left( -2^p x + 1 + 2^p D(\omega) \right) \text{1}_{\{(\omega, x) \in V_p\}},
\]

(iii) for any \( (\omega, x) \in \Omega \times \mathbb{R} \),
\[
\hat{\theta}_p(\omega, x) = \left( \theta_\omega, [x + f_p(\omega, x)\sigma(\omega) - \xi(\omega)]^+ \right).
\]

It is then easily checked, that for all \( p \), \( V_p \) is an open set, \( \hat{\theta} = \hat{\theta}_p \) outside \( V_p \), and that \( \hat{\theta}_p \) is continuous from \( \omega \times \mathbb{R} \) into itself.

\[
\begin{align*}
&\text{A path of } x \mapsto f_p(\omega, x) \\
&\downarrow \quad \downarrow \quad \downarrow \\
&0 \quad D(\omega) \quad D(\omega) + 2^{-p}
\end{align*}
\]
Let us first assume that condition (2.21) holds. Note, that for any \( i \geq 1 \), \( W_0^i \circ \theta^{-i} \) can be interpreted as the workload in the system at time 0 assuming that \( C_{-i} \) finds an empty system upon arrival. It is then easily checked by induction at the construction points (the instants in which a customer enters an empty system) that for all \( i \geq 1 \),

\[
W_0^i \circ \theta^{-i} \in \mathcal{I}, \quad P^0 \text{ - a.s.} \tag{2.22}
\]

Fix \( p \geq 1 \). As a consequence of (2.22), we have for any \( i \geq 1 \) that

\[
P^0 \left[ W_0^i \circ \theta^{-i} \in (D, D + 2^{-p}) \right]
\leq P^0 \left[ \min \left\{ \mathcal{I} \cap (D, D + 1) \right\} \in (D, D + 2^{-p}) \right] \cap \left\{ \mathcal{I} \cap (D, D + 1) \neq \emptyset \right\},
\]

and therefore

\[
\lim_{p \to \infty} \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} P^0 \left[ W_0^i \circ \theta^{-i} \in (D, D + 2^{-p}) \right]
\leq \lim_{p \to \infty} P^0 \left[ \min \left\{ \mathcal{I} \cap (D, D + 1) \right\} - D \in (0, 2^{-p}) \right] \cap \left\{ \mathcal{I} \cap (D, D + 1) \neq \emptyset \right\}
= 0. \tag{2.23}
\]

All the same, whenever (2.8) holds, in view of Theorem 2 there exists \( P^0 \)-a.s. a finite \( \tau \) such that \( W_0^i \circ \theta^{-i} = W \) for all \( i \geq \tau \). Hence, the random set

\[
\mathcal{E} = \left\{ W_0^i \circ \theta^{-i}; i \geq 1 \right\} = \left\{ W_0^i \circ \theta^{-i}; i \leq \tau \right\}
\]

is a.s. of finite cardinal, so that (2.23) holds replacing \( \mathcal{I} \) by \( \mathcal{E} \).

As a conclusion, assumption (A3) p.272 in [2] is verified under both conditions (2.21) and (2.8). Hence, Theorem 1 in [1] holds, yielding that there exists a \( \tilde{\theta} \)-invariant probability \( \tilde{P}^0 \) on \( \Omega \times \mathbb{R} \) whose \( \Omega \)-marginal is \( P^0 \).

It is now straightforward that

\[
\tilde{W} \circ \tilde{\theta}(\omega, x) = \varphi[\omega](x) = \tilde{\varphi}[\omega, x] \left( \tilde{W}(\omega, x) \right), \quad \tilde{P}^0 - \text{a.s.,}
\]

so that for all \( n \in \mathbb{N} \), \( \tilde{P}^0 \)-a.s.,

\[
\tilde{W}_{n+1} = \tilde{\varphi} \left[ \tilde{\theta}^n \right] \left( \tilde{W}_n \right),
\]

where \( \left\{ \tilde{W}_n, \tilde{\varphi} \left[ \tilde{\theta}^n \right] \right\} \) \( n \in \mathbb{N} \) is stationary under \( \tilde{P}^0 \).
Remark, that condition (2.21) typically holds whenever $\sigma$ and $\xi$ are valued in a lattice $\{xn; n \in \mathbb{N}\}$, where $x > 0$.

On another hand, note that (2.21) does not entail (2.8): take $\sigma(\omega_1) = 3$, $\sigma(\omega_2) = 1$, $D(\omega_1) = 3$ and $D(\omega_2) = 2$ in the first example of subsection 2.5. Then, (2.21) holds for $\mathcal{I} = \mathbb{N}$ whereas (2.8) is violated: there exists a weak stationary solution, but no stationary solution on the original probability space.

**Loss system.** Consider a loss system $G/G/1/1$: there is no buffer, so that each customer is served if and only if he finds an empty system upon arrival. This corresponds to a $G/G/1/1+G(b)$ queue in which the generic patience $D$ is null, $\mathbb{P}^0$-a.s.. The sufficient condition of existence of a stationary workload, constructed in [4], section 2.6, naturally corresponds to (2.8), taking $D \equiv 0$.

A stationary workload for that queue always exists on $\Omega \times \mathbb{R}$, as readily follows from Theorem 3, since

$$E \subset \left\{ \left[ \sigma - \sum_{i=1}^{i} \xi - j \right] ; i \geq 1 \right\},$$

which has $\mathbb{P}^0$-a.s. a finite cardinal since the sequence $\left\{ \sigma - \sum_{i=1}^{i} \xi - j \right\} i \geq 1$ tends $\mathbb{P}^0$-a.s. to $-\infty$, in view of Birkhoff’s Theorem. This result is proven in a similar way in [2], whereas an explicit construction of a stationary workload on $\Omega \times \mathbb{N}$ is proposed in [9] and [13] (see as well [12] for a generalization of that result to dominated lattice-valued SRS).

### 3. Impatience until the end of service

Let us now consider a $G/G/s/s+G(e)$ queue: the model is that of the previous sections, except that the customers are now assumed to remain impatient until the end of their service. Indeed, they leave the system, and are eliminated forever, provided that their service is not completed before their deadline. Keeping the notations and other assumptions of the previous section, customer $C_n$ is thus discarded when the total time he has to wait in the buffer and spend in the service booth is larger than his initial time credit $D_n$. We assume that the customers are unaware of their waiting time and deadline, and consequently wait in the system, and possibly enter service, as long as their deadline is not reached.
3.1. Construction points

For any $n \in \mathbb{Z}$ the maximal sojourn time of $C_n$ in the system is given by $D_n$, whereas its minimal sojourn time is $\sigma_n \land D_n$. Then, the LRmST sequence is that of the system with impatience until the beginning of service, whereas the LRMST sequence \( \{L_n\}_{n \in \mathbb{N}} \) is driven by the recursive equation

\[
L_{n+1} = [L_n \lor D_n - \xi_n]^+ = F_{D_n, \xi_n}(L_n)
\]

and in view of Lemma 1, the unique stationary LRMST reads

\[
Y_{D, \xi} = \left[ \sup_{j \in \mathbb{N}^*} \left( D_{-j} - \sum_{i=1}^{j} \xi_{-i} \right) \right]^+. \tag{3.1}
\]

As in Theorem 1, we have the following result.

**Theorem 4.** The $G/G/s+s+G(b)$ queue empties $\mathbb{P}^0$-a.s. an infinite number of times if

\[
\mathbb{P}^0 \left[ \sup_{j \in \mathbb{N}^*} \left( D_{-j} - \sum_{i=1}^{j} \xi_{-i} \right) \leq 0 \right] > 0, \tag{3.2}
\]

and only if (2.9) holds.

3.2. Single-server FIFO queue

Suppose now that the discipline is FIFO. The patience of some customer may finish while he is in service. Hence, such customers contribute to the workload, since some service is provided to them, whereas their service is not eventually completed. More precisely, the quantity of work added to the workload $W_n$ at the arrival of customer $C_n$ is given by

\[
\begin{cases}
\sigma_n & \text{if } W_n \leq (D_n - \sigma_n)^+,
\
\sigma_n - (W_n + \sigma_n - D_n) = D_n - W_n & \text{if } (D_n - \sigma_n)^+ < W_n \leq D_n,
\
0 & \text{if } W_n > D_n.
\end{cases}
\]

This can be reformulated in a compact form, stating that the workload sequence is driven by the recursive equation

\[
W_{n+1} = \left[ W_n + \sigma_n - (W_n + \sigma_n - D_n)^+ \right]^+ - \xi_n].^+
\]
Therefore, a stationary workload is a $\mathbb{R}_+$-valued r.v. $S$ that solves the equation
\[
S \circ \theta = \psi(S) := \left[S + \left(\sigma - (S + \sigma - D)^+\right)^+ - \xi\right]^+.
\] (3.3)

We have the following result.

**Theorem 5.** (i) The equation (3.3) admits a $\mathbb{P}^0$-a.s. finite solution $S$ that is such that
\[
Y_{\sigma \wedge D, \xi} \leq S \leq Y_{D, \xi}, \mathbb{P}^0 \text{-a.s.}
\]
(ii) If (3.2) holds, this solution is unique and for any r.v. $Z$ such that $Z \leq Y_{D, \xi}$, $\mathbb{P}^0$-a.s., $\{W^n\}_{n \in \mathbb{N}}$ converges with strong backwards coupling to $S$.
(iii) If in addition (2.8) holds, then the unique solution is such that $S \leq W$, $\mathbb{P}^0$-a.s., where $W$ is the only solution of (2.11).

**Proof.** (i) The random mapping $\psi$ is $\mathbb{P}^0$-a.s. non-decreasing and continuous, as easily checked. Hence a minimal solution $S$ to (3.3) can be constructed using Loynes’ Theorem. Let us now remark that for any $x$,
\[
\psi(x) = \left[\left((x + \sigma) \wedge D\right) \mathbb{1}_{\{x \leq D\}} + x \mathbb{1}_{\{x > D\}} - \xi\right]^+ \\
\leq \left[(x \vee D) \wedge (x + \sigma) \mathbb{1}_{\{x \leq D\}} - \xi\right]^+ = \varphi(x) \wedge F_{D, \xi}(x), \mathbb{P}^0\text{-a.s..} \quad (3.4)
\]
This clearly implies that the event $\{S \leq Y_{D, \xi}\}$ is $\theta$-contracting. On the other hand, $S$ is such that $\mathbb{P}^0[S \leq D] > 0$, since the contrary would imply that $S \circ \theta = S - \xi$, $\mathbb{P}^0$-a.s., a contradiction to the Ergodic Lemma. But on $\{S \leq D\}$,
\[
S \circ \theta = \left[(S + \sigma) \wedge D\right] - \xi^+ \leq [D \vee Y_{D, \xi} - \xi]^+ = Y_{D, \xi} \circ \theta.
\]
Consequently, we have that
\[
S \leq Y_{D, \xi}, \mathbb{P}^0 \text{-a.s.}
\]
Now, for any $x$ we also have that $\mathbb{P}^0$-a.s.,
\[
F_{\sigma \wedge D, \xi}(x) = \left[(D \wedge \sigma) \mathbb{1}_{\{x \leq D \wedge \sigma\}} + x \mathbb{1}_{\{x > D \wedge \sigma\}} \mathbb{1}_{\{x \leq D\}} + x \mathbb{1}_{\{x > D\}} - \xi\right]^+ \\
\leq \left[(D \wedge (x + \sigma)) \mathbb{1}_{\{x \leq D \wedge \sigma\}} + (D \wedge (x + \sigma)) \mathbb{1}_{\{x > D \wedge \sigma\}} \mathbb{1}_{\{x \leq D\}} + x \mathbb{1}_{\{x > D\}} - \xi\right]^+ \\
= \psi(x), \quad (3.5)
\]
which implies that the event \( \{ Y_{\sigma \wedge D, \xi} \leq S \} \) is \( \theta \)-contracting. Assuming that \( Y_{\sigma \wedge D, \xi} > \sigma \wedge D, \mathbb{P}^0 \)-a.s. would again contradict the Ergodic Lemma. Thus \( \{ Y_{\sigma \wedge D, \xi} \leq \sigma \wedge D \} \) is \( \mathbb{P}^0 \)-almost sure since on \( \{ Y_{\sigma \wedge D, \xi} \leq \sigma \wedge D \} \),

\[
Y_{\sigma \wedge D, \xi} \circ \theta = [\sigma \wedge D - \xi]^+ \leq \left[ (S + \sigma) \wedge D \right]_{\{S \leq D\}} + S 1_{\{S > D\}} - \xi^+ = S \circ \theta.
\]

(ii) For any solution \( S' \) of (3.3), \( \{ S' \leq S \} \) is \( \theta \)-contracting. This event is thus \( \mathbb{P}^0 \)-almost sure whenever (3.2) holds since it is included in \( \{ Y_{D, \xi} = 0 \} \). Hence the uniqueness of the solution is entailed by the minimality of \( S \).

On the other hand, (3.4) implies with a simple induction that

\[
W_n^Z \leq L_n^{Y_{D, \xi}} = Y_{D, \xi} \circ \theta^n, \quad n \in \mathbb{N},
\]

whenever \( Z \leq Y_{D, \xi} \). Thus, for any r.v. \( Z \) such that \( Z \leq Y_{D, \xi} \), \( \mathbb{P}^0 \)-a.s., the sequence \( \{ \{ Y_{D, \xi} \circ \theta^n = 0 \} \}_{n \in \mathbb{N}} \) is a sequence of renovating events of length 1 for \( \{ W_n^0 \}_{n \in \mathbb{N}} \). The strong backwards coupling property then follows, as in the proof of Theorem 2.

(iii) The fact that \( \psi \) is \( \mathbb{P}^0 \)-a.s. non-decreasing implies together with (3.4) that on the event \( \{ S \leq W \} \),

\[
S \circ \theta \leq \left[ W + \left( \sigma - (W + \sigma - D)^+ \right)^+ - \xi^+ \right]^+ \leq W \circ \theta.
\]

Therefore, \( \{ S \leq W \} \) is \( \theta \)-contracting. This event is \( \mathbb{P}^0 \)-almost sure whenever (2.8) holds since it includes \( \{ Y_{D+\sigma, \xi} = 0 \} \).

We represent hereafter a sample path of the random functions \( \varphi, \psi, F_{\sigma \wedge D}, F_D \) and \( F_{\sigma + D} \) to illustrate the comparisons (2.13), (2.16), (3.4) and (3.5). Note in particular that \( \psi \) is continuous, whereas \( \varphi \) is not.
For this model, the stationary loss probability $\pi(e)$ is the probability that the patience of customer $C_0$ is less than the sum of the stationary workload and his service time, i.e.

$$\pi(e) = P_S > D - \sigma.$$ 

From (i) of Theorem 5, we have that

$$P_{Y_{\sigma \wedge D, \xi}} > D - \sigma \leq \pi(e) \leq P_{Y_{D, \xi}} > D - \sigma.$$ 

On the other hand, the stationary probability $\hat{\pi}(e)$ that a customer does not reach the server is given by

$$\hat{\pi}(e) = P_S > D.$$ 

Then in view of (2.17) and (iii) of Theorem 5, the loss probability $\pi(b)$ of $G/G/1/1+G(b)$ is larger than $\hat{\pi}(e)$ for the same parameters.
References


