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SEMI–MARKOV PROCESSES AND HIDDEN MODELS

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Abstract: The purpose of this article is to present the semi-Markov processes focusing on applications, especially in reliability and dependability. The under consideration semi-Markov processes are both of continuous and discrete time with countable or finite state space. The basic definitions of Markov renewal and semi-Markov processes are presented, as well as the Markov renewal theorem and the necessary theory of statistical estimation. In the sequel, we describe a general reliability model and give the corresponding estimations. Finally, we briefly present the semi-Markov chain and hidden semi-Markov models. Some bibliographical directions for further relevant topics, which are not included in this article, are given.

Keywords and phrases: Semi-Markov Process, Markov Renewal Process, Semi-Markov chain, Hidden Semi-Markov Model, Semi-Markov Kernel, Markov Renewal Equation, Empirical Estimator, Reliability, Availability, Mean Time To Failure.

1 MARKOV RENEWAL AND SEMI–MARKOV PROCESSES

Semi-Markov processes were introduced independently by W.L. Smith and P. Lévy in the same Mathematical meeting in 1954. Semi-Markov processes constitute a generalization of Markov and renewal processes. Jump Markov processes, Markov chains, renewal processes - ordinary, alternating, delayed, and stopped - are particular cases of semi-Markov processes. As Feller [13] pointed out, the basic theory of semi-Markov processes was introduced by Pyke [41, 42]. Further significant results are obtained by Pyke and Schaufele [43, 44], Çinlar [9, 10], Koroliuk [25, 26, 27], and many others. Also, see [11, 14]. Currently, semi-Markov processes have achieved significant importance in probabilistic and statistical modeling. Our main references for this article are [29, 6, 37].

Consider an infinite countable set, say E, and an E-valued pure jump stochastic process $Z = (Z_t)_{t\in\mathbb{R}_+}$. Let $0 = S_0 \leq S_1 \leq ... \leq S_n \leq S_{n+1} \leq ...$ be the jump times of Z, and $J_0, J_1, J_2, \dots$ the successive visited states of Z. Note that S_0 may also take positive values. Let $\mathbb{N}$ be the set of nonnegative integers.

Definition 1. *The stochastic process $(J_n, S_n)_{n\in\mathbb{N}}$ is said to be a Markov renewal process (MRP), with state space E, if it satisfies a.s., the following equality*

$$\mathbb{P}(J_{n+1} = j, S_{n+1} - S_n \leq t \mid J_0, \ldots, J_n; S_1, \ldots, S_n) = \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n \leq t \mid J_n)$$

for all $j \in E$, all $t \in \mathbb{R}_+$ and all $n \in \mathbb{N}$. In this case, Z is called a semi-Markov process (SMP).

We assume that the above probability is independent of n and S_n, and in this case the MRP is called *time homogeneous*. The MRP $(J_n, S_n)_{n \in \mathbb{N}}$ is determined by the transition kernel $Q_{ij}(t) := \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n \leq t \mid J_n = i)$, called the *semi-Markov kernel*, and the *initial distribution* α, with $\alpha(i) = \mathbb{P}(J_0 = i)$, $i \in E$. The process (J_n) is a Markov chain with state space E and transition probabilities $P(i,j) := Q_{ij}(\infty) := \lim_{t \to \infty} Q_{ij}(t)$, called the embedded Markov chain (EMC) of Z. It is worth noticing that here $Q_{ii}(t) \equiv 0$, for all $i \in E$, but in general we can consider semi-Markov kernels by dropping this hypothesis.

The semi-Markov process Z is connected to (J_n, S_n) by

$$Z_t = J_n, \quad \text{if} \quad S_n \leq t < S_{n+1}, \quad t \geq 0 \qquad \text{and} \quad J_n = Z_{S_n}, \quad n \geq 0.$$

A Markov process with state space $E = \mathbb{N}$ and generating matrix $A = (a_{ij})_{i,j \in E}$ is a special semi-Markov process with semi-Markov kernel

$$Q_{ij}(t) = \frac{a_{ij}}{a_i}(1 - e^{-a_i t}), \quad i \neq j, \quad a_i \neq 0,$$

where $a_i := -a_{ii}$, $i \in E$, and $Q_{ij}(t) = 0$, if $i = j$ or $a_i = 0$.

Also, define $X_n := S_n - S_{n-1}$, $n \geq 1$, the inter–jump times, and the process $(N(t))_{t \in \mathbb{R}_+}$, which counts the number of jumps of Z in the time interval $(0, t]$, by $N(t) := \sup\{n \geq 0 : S_n \leq t\}$. Also, define $N_i(t)$ to be the number of visits of Z to state $i \in E$ in the time interval $(0, t]$. To be specific,

$$N_i(t) := \sum_{n=1}^{N(t)} \mathbf{1}_{\{J_n = i\}} = \sum_{n=1}^{\infty} \mathbf{1}_{\{J_n = i, S_n \leq t\}}, \quad \text{and also} \quad N_i^*(t) = \mathbf{1}_{\{J_0 = i, S_0 \leq t\}} + N_i(t).$$

If we consider the (eventually delayed) renewal process $(S_n^i)_{n \geq 0}$ of successive times of visits to state i, then $N_i(t)$ is the counting process of renewals. Denote by μ_{ii} the mean recurrence time of (S_n^i), i.e., $\mu_{ii} = \mathbb{E}[S_2^i - S_1^i]$.

Let us denote by $Q(t) = (Q_{ij}(t),\ i,j \in E)$, $t \geq 0$, the semi–Markov kernel of Z. Then we can write:

$$Q_{ij}(t) := \mathbb{P}(J_{n+1} = j, X_{n+1} \leq t \mid J_n = i) = P(i,j) F_{ij}(t), \quad t \geq 0, \quad i,j \in E, \quad (1)$$

where $P(i,j) := \mathbb{P}(J_{n+1} = j \mid J_n = i)$ is the transition kernel of the EMC (J_n), and $F_{ij}(t) := \mathbb{P}(X_{n+1} \leq t \mid J_n = i, J_{n+1} = j)$ is the conditional distribution function of the sojourn time in the state i given that the next visited state is j, $(j \neq i)$. Let us also, define the distribution function $H_i(t) := \sum_{j \in E} Q_{ij}(t)$ and its mean value m_i, which is the mean sojourn time of Z in state i. In general, Q_{ij} is a subdistribution, i.e., $Q_{ij}(\infty) \leq 1$, hence H_i is a distribution function, $H_i(\infty) = 1$, and $Q_{ij}(0-) = H_i(0-) = 0$.

A special case of semi-Markov processes is the one where $F_{ij}(\cdot)$ does not depend on j, i.e., $F_{ij}(t) \equiv F_i(t) \equiv H_i(t)$, and

$$Q_{ij}(t) = P(i,j) F_i(t). \quad (2)$$

Any general semi-Markov process can be transformed into one of the form (2), (see, e.g., [29]).

Let $\phi(i,t),\ \ i \in E, t \geq 0$, be a real-valued measurable function and define the convolution of ϕ by Q as follows

$$Q * \phi(i,t) := \sum_{k \in E} \int_0^t Q_{ik}(ds)\phi(k, t-s). \tag{3}$$

Now, consider the n-fold convolution of Q by itself. For any $i,j \in E$,

$$Q_{ij}^{(n)}(t) = \begin{cases} \sum_{k\in E} \int_0^t Q_{ik}(ds) Q_{kj}^{(n-1)}(t-s) & n \geq 2 \\ Q_{ij}(t) & n = 1 \\ \delta_{ij} \mathbf{1}_{\{t \geq 0\}} & n = 0. \end{cases}$$

It is easy to prove (e.g., by induction) the following fundamental equality

$$Q_{ij}^{(n)}(t) = \mathbb{P}_i(J_n = j, S_n \leq t). \tag{4}$$

Here, as usual, $\mathbb{P}_i(\cdot)$ means $\mathbb{P}(\cdot \mid J_0 = i)$, and $\mathbb{E}_i$ is the corresponding expectation.

Let us define the Markov renewal function $\psi_{ij}(t),\ i,j \in E, t \geq 0$, by

$$\begin{aligned} \psi_{ij}(t) &:= \mathbb{E}_i[N_j^*(t)] = \mathbb{E}_i \sum_{n=0}^{\infty} \mathbf{1}_{\{J_n = j, S_n \leq t\}} \\ &= \sum_{n=0}^{\infty} \mathbb{P}_i(J_n = j, S_n \leq t) = \sum_{n=0}^{\infty} Q_{ij}^{(n)}(t). \end{aligned} \tag{5}$$

Another important function is the semi-Markov transition function

$$P_{ij}(t) := \mathbb{P}(Z_t = j \mid Z_0 = i), \quad i,j \in E, t \geq 0,$$

which is the conditional marginal law of the process. We will study this function in the next section.

Definition 2. *The semi-Markov process Z is said to be regular if*

$$\mathbb{P}_i(N(t) < \infty) = 1,$$

for any $t \geq 0$ and any $i \in E$.

For regular semi-Markov processes we have $S_n < S_{n+1}$, for any $n \in \mathbb{N}$, and $S_n \to \infty$. In the sequel, we are concerned with regular semi-Markov processes. The following theorem gives two criteria for regularity.

Theorem 1. *A semi-Markov process is regular if one of the following conditions is satisfied:*

(1) ([41]) for every sequence $(j_0, j_1, ...) \in E^{\infty}$ and every $C > 0$, at least one of the series

$$\sum_{k \geq 0} [1 - F_{j_k j_{k+1}}(C)], \qquad \sum_{k \geq 0} \int_0^C t F_{j_k j_{k+1}}(dt),$$

diverges;

(2) ([46]) there exist constants, say $\alpha > 0$ and $\beta > 0$, such that $H_i(\alpha) < 1 - \beta$, for all $i \in E$.

Let us now discuss the nature of different states of an MRP. An MRP is irreducible, if, and only if, its EMC (J_n) is irreducible. A state i is recurrent (transient) in the MRP, if, and only if, it is recurrent (transient) in the EMC. For an irreducible finite MRP, a state i is positive recurrent in the MRP, if, and only if, it is recurrent in the EMC and if for all $j \in E$, $m_j < \infty$. If the EMC of an MRP is irreducible and recurrent, then all the states are positive-recurrent, if, and only if, $m := \nu m := \sum_i \nu_i m_i < \infty$, and null-recurrent, if, and only if, $m = \infty$ (where ν is the stationary probability of EMC (J_n)). A state i is said to be periodic with period $a > 0$ if $G_{ii}(\cdot)$ (the distribution function of the random variable $S_2^i - S_1^i$) is discrete concentrated on $\{ka : k \in \mathbb{N}\}$. Such a distribution is said to be also periodic. In the opposite case it is called aperiodic. Note that the term *period* has a completely different meaning from the corresponding one of the classic Markov chain theory.

2 MARKOV RENEWAL THEORY

As the renewal equation in the case of the renewal process theory on the half-real line, the Markov renewal equation is a basic tool in the theory of semi-Markov processes.

Let us write the Markov renewal function (5) in matrix form

$$\psi(t) = (I(t) - Q(t))^{(-1)} = \sum_{n=0}^{\infty} Q^{(n)}(t). \tag{6}$$

This can also be written as

$$\psi(t) = I(t) + Q * \psi(t), \tag{7}$$

where $I(t) = I$ (the identity matrix), if $t \geq 0$ and $I(t) = 0$, if $t < 0$.

Equation (7) is a special case of what is called a *Markov Renewal Equation* (MRE). A general MRE is as follows

$$\Theta(t) = L(t) + Q * \Theta(t), \tag{8}$$

where $\Theta(t) = (\Theta_{ij}(t))_{i,j\in E}$, $L(t) = (L_{ij}(t))_{i,j\in E}$ are matrix-valued measurable functions, with $\Theta_{ij}(t) = L_{ij}(t) = 0$ for $t < 0$. The function $L(t)$ is a given matrix-valued function and $\Theta(t)$ is an unknown matrix-valued function. We may also consider a vector version of Equation (8), i.e., consider corresponding columns of the matrices Θ and L.

Let $\mathbf{B}$ be the space of all locally bounded, on $\mathbb{R}_+$, matrix functions $\Theta(t)$, i.e., $\|\Theta(t)\| = \sup_{i,j} |\Theta_{i,j}(t)|$ is bounded on sets $[0, \xi]$, for every $\xi \in \mathbb{R}_+$. Also, denote by $\overline{H}_i(t) := 1 - H_i(t)$.

Theorem 2. *(Markov Renewal Theorem [47]) Let the following conditions be fulfilled:*

(1) The EMC (J_n) is ergodic, i.e., irreducible and positive recurrent, with stationary probability $\nu = (\nu_i, i \in E)$.

(2) The mean sojourn time in every state is finite, i.e., for every $i \in E$,

$$m_i := \int_0^\infty \overline{H}_i(t)dt < \infty, \quad \textit{and} \quad m := \sum_{i\in E} \nu_i m_i > 0.$$

(3) The distribution functions $H_i(t)$, $i \in E$, are non periodic.

(4) The functions $L_{ij}(t), t \geq 0$, are direct Riemann integrable, i.e., they satisfy the following two conditions, for any $i, j \in E$:

$$\sum_{n \geq 0} \sup_{n \leq t \leq n+1} |L_{ij}(t)| < \infty,$$

and

$$\lim_{\Delta \downarrow 0} \left\{ \Delta \sum_{n \geq 0} \left[\sup_{n\Delta \leq t \leq (n+1)\Delta} L_{ij}(t) - \inf_{n\Delta \leq t \leq (n+1)\Delta} L_{ij}(t) \right] \right\} = 0.$$

Then Equation (8) has a unique solution $\Theta = \psi * L(t)$ belonging to $\mathbf{B}$, and

$$\lim_{t \to \infty} \Theta_{ij}(t) = \frac{1}{m} \sum_{\ell \in E} \nu_\ell \int_0^\infty L_{\ell j}(t) dt. \tag{9}$$

The following result is an important application of the above theorem.

Proposition 1. *The transition function $P(t) = (P_{ij}(t))$ satisfies the following MRE*

$$P(t) = I(t) - H(t) + Q * P(t),$$

which, under Conditions (1)-(3) of Theorem 2, has the unique solution

$$P(t) = \psi * (I(t) - H(t)), \tag{10}$$

and, for any $i, j \in E$,

$$\lim_{t \to \infty} P_{ji}(t) = \nu_i m_i / m =: \pi_i. \tag{11}$$

Here $H(t) = diag(H_i(t))$ is a diagonal matrix.

It is worth noticing that, in general, the stationary distribution π of the semi-Markov process Z is not equal to the stationary distribution ν of the embedded Markov chain (J_n). Nevertheless, we have $\pi = \nu$ when, for example, m_i is independent of $i \in E$.

3 STATISTICAL INFERENCE

Statistical inference for semi-Markov processes is provided in several papers. Moore and Pyke [33] studied empirical estimators for finite semi-Markov kernels; Lagakos, Sommer and Zelen [28] gave maximum likelihood estimators for nonergodic finite semi-Markov kernels; Akritas and Roussas [2] gave parametric local asymptotic normality results for semi-Markov processes; Gill [15] studied Kaplan-Meier type estimators by point process theory; Greenwood and Wefelmeyer [17] studied efficiency of empirical estimators for linear functionals in the case of a general state space; Ouhbi and Limnios [36] studied nonparametric estimators of semi-Markov kernels, non-linear functionals of semi-Markov kernels, including Markov renewal matrices and reliability functions [37], and rate of occurrence of failure functions [38]. Also see the book [1]. We will give here some elements of the nonparametric estimation of semi-Markov kernels.

Let us consider an observation of an irreducible semi-Markov process Z, with finite state space E, up to a fixed time T, i.e., $(Z_s, 0 \le s \le T) \equiv (J_0, J_1, ..., J_{N(T)}; X_1, ...$ $..., X_{N(T)-1}, T - S_{N(T)})$, if $N(T) > 0$ and $(Z_s, 0 \le s \le T) \equiv (J_0)$ if $N(T) = 0$.

The empirical estimator $\widehat{Q}_{ij}(t,T)$ of $Q_{ij}(t)$ is defined by

$$\widehat{Q}_{ij}(t,T) = \frac{1}{N_i(T)} \sum_{k=1}^{N(T)} \mathbf{1}_{\{J_{k-1}=i, J_k=j, X_k \le t\}}. \tag{12}$$

Then we have the following result from Moore and Pyke [33], see also [36]. We donote by $\xrightarrow{a.s.}$ and $\xrightarrow{d}$ the almost sure convergence and convergence in distribution respectively. Denote by $N(a,b)$ the normal random variable with mean a and variance b.

Theorem 3. *For any fixed $i, j \in E$, as $T \to \infty$, we have:*

(a) (Strong consistency) $\max_{i,j} \sup_{t\in(0,T)} \left|\widehat{Q}_{ij}(t,T) - Q_{ij}(t)\right| \xrightarrow{a.s.} 0,$

(b) (Asymptotic normality) $T^{1/2}\left(\widehat{Q}_{ij}(t,T) - Q_{ij}(t)\right) \xrightarrow{d} N(0, \sigma_{ij}^2(t)),$

where $\sigma_{ij}^2(t) := \mu_{ii} Q_{ij}(t)[1 - Q_{ij}(t)]$.

We define estimators of the Markov renewal function and of transition probabilities by plug in procedure, i.e., replacing semi-Markov kernel $Q_{ij}(t)$ in Equations (6) and (10) by the empirical estimator kernel $\widehat{Q}_{ij}(t,T)$.

The following two results concerning the Markov renewal function estimator, $\widehat{\psi}_{ij}(t,T)$, and the transition probability function estimator, $\widehat{P}_{ij}(t,T)$, of Z were proved by Ouhbi and Limnios [36].

Theorem 4. ([36]). *The estimator $\widehat{\psi}_{ij}(t,T)$ of the Markov renewal function $\psi_{ij}(t)$ satisfies the following two properties:*

(a) (Strong consistency) it is uniformly strongly consistent, i.e., as $T \to \infty$,

$$\max_{i,j} \sup_{t\in(0,T)} \left|\widehat{\psi}_{ij}(t,T) - \psi_{ij}(t)\right| \xrightarrow{a.s.} 0.$$

(b) (Asymptotic normality) For any fixed $t > 0$, it converges in distribution, as $T \to \infty$, to a normal random variable, i.e.,

$$T^{1/2}(\widehat{\psi}_{ij}(t,T) - \psi_{ij}(t)) \xrightarrow{d} N(0, \sigma_{ij}^2(t)),$$

where $\sigma_{ij}^2(t) = \sum_{r\in E}\sum_{k\in E} \mu_{rr}\{(\psi_{ir} * \psi_{kj})^2 * Q_{rk} - (\psi_{ir} * \psi_{kj} * Q_{rk})^2\}(t)$.

Theorem 5. ([36]). *The estimator $\widehat{P}_{ij}(t,T)$ of the transition function $P_{ij}(t)$, satisfies the following two properties:*

(a) (Strong consistency) for any fixed $L > 0$, we have, as $T \to \infty$

$$\max_{i,j} \sup_{t\in[0,L]} \left|\widehat{P}_{ij}(t,T) - P_{ij}(t)\right| \xrightarrow{a.s.} 0;$$

(b) (Asymptotic normality) For any fixed $t > 0$, we have, as $T \to \infty$,

$$T^{1/2}(\widehat{P}_{ij}(t,T) - P_{ij}(t)) \xrightarrow{d} N(0, \sigma_{ij}^2(t)),$$

where

$$\sigma_{ij}^2(t) = \sum_{r\in E}\sum_{k\in E}\mu_{rr}[(1-H_i)*B_{irkj} - \psi_{ij}\mathbf{1}_{\{r=j\}}]^2 * Q_{rk}(t)$$
$$- \{[(1-H_i)*B_{irkj} - \psi_{ij}\mathbf{1}_{\{r=j\}}] * Q_{rk}(t)\}^2,$$

and

$$B_{irkj}(t) = \sum_{n=1}^{\infty}\sum_{\ell=1}^{n} Q_{ir}^{(\ell-1)} * Q_{kj}^{(n-\ell)}(t).$$

4 RELIABILITY OF SEMI-MARKOV SYSTEMS

For a stochastic system with state space E described by a semi-Markov process (Z_t), let us consider a partition U, D of E, i.e., $E = U \cup D$, with $U \cap D = \emptyset$, $U \neq \emptyset$, and $D \neq \emptyset$. The set U contains the up states and D contains the down states of the system. The reliability is defined by $R(t) := \mathbb{P}(Z_s \in U, \forall s \in [0,t])$. If we define the conditional reliability by $R_i(t) := \mathbb{P}(Z_s \in U, \forall s \in [0,t] \mid Z_0 = i)$, then for any initial distribution law vector α, we have $R(t) := \sum_{i\in U}\alpha_i R_i(t)$. For the finite state space case, without loss of generality, let us enumerate first the up states and next the down states, i.e., for $E = \{1,2,...,d\}$, we have $U = \{1,...,r\}$ and $D = \{r+1,...,d\}$.

The conditional reliability $R_i(t)$ fulfills the following MRE

$$R_i(t) = \delta_{ij}\overline{H}_i(t) + \sum_{j\in U}\int_0^t Q_{ij}(ds)R_j(t-s), \tag{13}$$

which has the following solution

$$R_i(t) = \sum_{j\in U}(\psi_{ij} * \delta_{ij}\overline{H}_i)(t), \tag{14}$$

where $\delta_{ij} = 1$, if $i = j$ and $= 0$ otherwise.

Integrating (13) over $[0,\infty)$, we get

$$(MTTF_1, ..., MTTF_r)' = (I - P_{11})^{-1}(m_1, ..., m_r)', \tag{15}$$

where $MTTF_j$ is the mean time to failure starting from the state $j \in E$, and C' denotes the transpose of the vector or the matrix C.

The above reliability indicators and some additional ones are formulated in matrix form and are given in the following table.

Notation. In Table 1, we have $\overline{\mathbf{H}}_1(t) = (\overline{H}_i(t); i \in U)'$, $\overline{\mathbf{H}}_2(t) = (\overline{H}_i(t); i \in D)'$, $\overline{\mathbf{H}}_{10}(t) = (\overline{H}_{10}(t); i \in U; 0, ..., 0)'$, $\mathbf{m}_1 = (m_i; i \in U)'$, $\mathbf{m}_2 = (m_i; i \in D)'$, $\mathbf{1}_r = (1, ..., 1)'$ r-ones. All of the above vectors are column vectors. Furthermore, index 1 means restriction of the corresponding vector or matrix on U and index 2 means restriction of the corresponding vector or matrix on D, i.e., $Q_{11}(t)$, means restriction of the matrix $Q(t)$ on $U \times U$ and α_1 restriction of the row vector α on U. By $A^{(-1)}(t)$ we mean the inverse of the matrix A in the convolution sense (see [29]). For example, if

$$A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, \quad \text{then} \quad A^{(-1)}(t) = (a*d(t) - b*c(t))^{(-1)} * \begin{pmatrix} d(t) & -b(t) \\ -c(t) & a(t) \end{pmatrix}.$$

Table 1. Closed form solution for reliability and related measurements

Reliability	$R(t) = \alpha_1(\mathbf{I} - \mathbf{Q}_{11})^{(-1)} * \overline{\mathbf{H}}_1(t)$
Availability	$A(t) = \alpha(\mathbf{I} - \mathbf{Q})^{(-1)} * \overline{\mathbf{H}}_{10}(t)$
Maintainability	$M(t) = 1 - \alpha_2(\mathbf{I} - \mathbf{Q}_{22})^{(-1)} * \overline{\mathbf{H}}_2(t)$
Mean Time To Failure	$\text{MTTF} = \alpha_1(\mathbf{I} - \mathbf{P}_{11})^{-1}\mathbf{m}_1$
Mean Time To Repair	$\text{MTTR} = \alpha_2(\mathbf{I} - \mathbf{P}_{22})^{-1}\mathbf{m}_2$
Mean Up Time	$\text{MUT} = \frac{\nu_1 \mathbf{m}_1}{\nu_2 \mathbf{P}_{21}\mathbf{1}_r}$
Mean Down Time	$\text{MDT} = \frac{\nu_2 \mathbf{m}_2}{\nu_1 \mathbf{P}_{12}\mathbf{1}_{d-r}}$

Reliability estimation. From estimator (12), the following plug in estimator of reliability is proposed

$$\widehat{R}(t,T) = \widehat{\alpha}_1(\mathbf{I} - \widehat{\mathbf{Q}}_{11})^{(-1)} * \widehat{\overline{\mathbf{H}}}_1(t,T).$$

The following properties are fulfilled by the above reliability estimator.

Theorem 6. *([37]). For any fixed $t > 0$ and for any $L \in (0,\infty)$, we have*

(a) (Strong consistency)

$$\sup_{0 \le t \le L} \mid \widehat{R}(t,T) - R(t) \mid \xrightarrow{a.s.} 0, \quad as \quad T \to \infty,$$

(b) (Asymptotic normality)

$$T^{1/2}(\widehat{R}(t,T) - R(t)) \xrightarrow{d} N(0, \sigma_R^2(t)), \quad as \quad T \to \infty,$$

where

$$\sigma_R^2(t) = \sum_{i \in U}\sum_{j \in E} \mu_{ii}\{(B_{ij}\mathbf{1}_{\{j \in U\}} - \sum_{r \in U} \alpha(r)\psi_{ri})^2 * Q_{ij}(t)$$
$$-[(B_{ij}\mathbf{1}_{\{j \in U\}} - \sum_{r \in U} \alpha(r)\psi_{ri}) * Q_{ij}(t)]^2\}$$

and

$$B_{ij}(t) = \sum_{n \in E}\sum_{k \in U} \alpha(i)\psi_{ni} * \psi_{jk} * (I - H_k(t))).$$

For further results on estimation, see [36, 37, 38, 30, 31].

5 SEMI-MARKOV CHAINS AND HIDDEN MODELS

In this section, we present semi-Markov chains (SMC), which are semi-Markov processes in discrete-time, and hidden semi-Markov models (HSMM).

Let us consider the Markov renewal process $(J_n, S_n)_{n \in \mathbb{N}}$ in discrete time $k \in \mathbb{N}$, with state space the countable set E. The semi-Markov kernel q is defined by

$$q_{ij}(k) := \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n = k \mid J_n = i), \quad i,j \in E, \quad k,n \in \mathbb{N}. \tag{16}$$

Define also $q_{ij}(K) = \sum_{k\in K} q_{ij}(k)$, where $K \subset \mathbb{N}$. So, as in the continuous time case, a SMC is determined by its semi-Markov kernel q and its initial distribution law α.

The process (J_n) is the embedded Markov chain of the MRP (J_n, S_n) with transition kernel $P = (P(i,j))$. The semi-Markov kernel q is written as

$$q_{ij}(k) = P(i,j)f_{ij}(k),$$

where $f_{ij}(k) := \mathbb{P}(S_{n+1} - S_n = k \mid J_n = i, J_{n+1} = j)$, is the conditional distribution of the sojourn time in the state i given that the next visited state is j, $(j \neq i)$.

Define the counting process of jumps $N(k) := \max\{n \geq 0 : S_n \leq k\}$. The semi-Markov chain is defined as follows

$$Z_k = J_{N(k)}, \quad k \in \mathbb{N}. \tag{17}$$

When $q_{ij}(k) = p_{ij}(p_{ii})^{k-1}$, $k = 1, 2, ...$, for all $i, j \in E$, $i \neq j$, then the SMC Z is a Markov chain with transition matrix $p = (p_{ij})$. In this case, the transition matrix of the EMC is $P(i,j) = p_{ij}/(1 - p_{ii})$, if $p_{ii} < 1$ and $i \neq j$, and $P(i,j) = 0$, if $i = j$ or $p_{ii} = 1$, and the conditional transition function is $f_{ij}(k) = (1 - p_{ii})(p_{ii})^{k-1}$.

The Markov renewal function ψ is defined by

$$\psi_{ij}(k) := \sum_{n=0}^{k} q_{ij}^{(n)}(k). \tag{18}$$

The general Markov renewal equation in the discrete case is as follows

$$\Theta(k) = L(k) + q * \Theta(k),$$

where $\Theta(k), k \in \mathbb{N}$, is an unknown sequence of real matrices (of finite or of infinite dimension), $L(k), k \in \mathbb{N}$, is a known sequence of real matrices, and $q(k), k \in \mathbb{N}$, a semi-Markov kernel. The convolution in discrete time is $(q*\Theta)_{ij}(k) = \sum_{\ell\in E}\sum_{s=0}^{k} q_{i\ell}(s)\Theta_{\ell j}(k-s)$. The general solution of the above MRE is given by

$$\Theta(k) = \psi * L(k).$$

For example, the transition probabilities of the SMC Z, $P_{ij}(k)$ satisfy the following MRE in matrix form,

$$P(k) = I - H(k) + q * P(k),$$

where its unique solution is

$$P(k) = \psi * (I - H)(k).$$

The reliability indicators are given by the same formulae as in the continuous-time case, in Table 1, only replacing Q by q and the step function $I = I(t)$, by the Dirac matrix measure $I\delta$ at the origin (see [6]). Estimation results for discrete-time semi-Markov chains can be found in [6].

A hidden semi-Markov model (HSMM) is defined by a bivariate process (Z_k, Y_k), where Z_k is an E-valued semi-Markov chain (unobserved) and Y_k is an A-valued process (observed) depending on Z_k. We suppose here that E and A are countable sets. This dependence structure in the HSMM is given by the following relation

$$\mathbb{P}(Z_{k+1} = j, Y_{k+1} = l \mid Z_s, Y_s; s \in \{0, 1, ..., k\}) = \mathbb{P}(Z_{k+1} = j \mid Z_k, U_k) \times \mathbb{P}(Y_{k+1} = l \mid Z_{k+1} = j),$$

where $U_k = k - S_{N(k)}$. This is a SM-M0 type HSMM. A more general type is the SM-Mm HSMM, which satisfies the following relation

$$\mathbb{P}(Z_{k+1} = j, Y_{k+1} = l \mid Z_s, Y_s; s \in \{0, 1, ..., k\}) = \mathbb{P}(Z_{k+1} = j \mid Z_k, U_k) \times \mathbb{P}(Y_{k+1} = l \mid Z_{k+1} = j, Y_k, ..., Y_{k-m+1}).$$

The main goal here is to estimate the semi-Markov kernel and, of course, the conditional distributions of Y, by observing only Y over $[0, M]$, i.e., $(Y_0^M = y_0^M) \equiv (Y_0 = y_0, Y_1 = y_1, ..., Y_M = y_M)$, where $(y_0, y_1, ..., y_M) \in A^{M+1}$. The quantities to be estimated here are $\theta = (q, \Gamma)$, where q is the semi-Markov kernel, and Γ the conditional distribution of Y, i.e., $\Gamma(i, l) = \mathbb{P}(Y_k = l \mid Z_k = i)$ for any $k \in \mathbb{N}$, $i \in E$, $l \in A$.

The likelihood function for the complete data set is

$$L_M(\theta; Z_0^M, Y_0^M) = \alpha(Z_0) \prod_{n=1}^{N(M)} q(J_{n-1}, J_n, X_n) \overline{H}(J_{N(M)}, U_M) \prod_{k=0}^{M} \Gamma(Z_k, Y_k).$$

Here, for convenience, we used the notation $q(i, j, k)$ instead of $q_{ij}(k)$ for all functions.

The maximum likelihood estimators can be obtained by an algorithm for incomplete data such as EM or stochastic EM. In fact, instead of maximizing the likelihood of the observed data, i.e., $L_M^0(\theta; Y_0^M) = \sum_{Z_0^M \in E^{M+1}} L_M(\theta; Z_0^M, Y_0^M)$, one maximizes the conditional expectation over θ,

$$\mathbb{E}_{\theta^{(m)}} \left[\log L_M(\theta; Z_0^M, Y_0^M) \mid Y_0^M \right]$$

in order to obtain $\theta^{(m+1)}$, and so on. Under appropriate general conditions the sequence $\theta^{(m)}, m \geq 0$, converges to the true value of the parameter θ.

For a given system whose the temporary behavior is described by a SMC, say Z, and the values of some indicators, i.e., noise, temperature, pressure, etc., are observed, say Y, we would like to estimate the state of the system in a given present time or into the future by observing Y only. This is a typical hidden semi-Markov problem.

6 CONCLUDING REMARKS

Several topics concerning SMP have been omitted. For example, results on time-nonhomogeneous SMP can be found in [22, 23, 24]; SM decision processes [40, 46]; calculation and numerical aspects [7, 23, 29]; limit theorems and convergence of SMP in series scheme [5, 18, 19, 25, 26, 27, 32, 34]; stochastic approximations [26, 27]; (stochastic) additive functionals [29, 26, 27]; diffusion type SMP [20]; entropy of SMP [16], etc., and also some important applications, as reliability and maintenance theory [6, 12, 21, 29, 37, 38, 35, 39]; queuing theory [3, 4]; insurance and finance [23, 45, 27]; words occurrences [8]; estimation in discrete time [6, 48]; phase-type distributions [4, 29, 39], etc. Several chapters in [22, 23] concern the above aspects. For a detailed presentation of semi-Markov chains and hidden semi-Markov models see [6].

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