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Reliability Measures of Semi-Markov Systems with General State Space

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Abstract The aim of this paper is to present a systematic modeling of reliability and related measures: availability, maintainability, failure rate, rate of occurrence of failures, mean times, etc., known in the literature under the term *dependability*. This model includes the continuous and discrete time semi-Markov processes with general state space. This is one of the most general models in reliability theory since it includes as particular cases the Markov and renewal processes.

Keywords Semi-Markov process · Reliability · Availability · Failure rate · Rate of occurrence of failure · Mean time to failure · Survival function

AMS 2000 Subject Classifications 60K15 · 90B25

1 Introduction

Reliability and related measures, as availability, maintainability, failure rate, mean times, etc., known under the term *dependability*, are very important in design, development and life of real technical systems. From a mathematical point of view, the problems related to reliability are mostly concerned with the hitting time of a so called failed or down subset of states of the system (Barbu and Limnios 2008; Iosifescu et al. 2010; Keilson 1979; Korolyuk and Limnios 2004, 2005; Limnios and Oprişan 2001; Osaki 1985; Pyke 1961; Ravichandran 1990).

Hitting times are important problems in theory and applications of stochastic processes. We encounter their distributions as reliability function in technical systems, or survival function in biology and medicine, or ruin time distribution in insurance and finance, etc. In this paper we are mostly concerned with reliability, but

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for the other above mentioned problems one can use the same models and formulae presented in this paper. For general results on hitting times in semi-Markov setting (see, e.g., Silvestrov 1996, Korolyuk and Limnios 2005, Iosifescu et al. 2010).

Reliability is concerned with the time to failure of a system which is the time to hitting the down set of states in the multi-state systems reliability theory (Aven and Jensen 1999; Barlow and Prochan 1975; Kovalenko et al. 1997; Lisnianski and Levitin 2003; Wilson et al. 2005). In the case of repairable systems this is the time to the first failure. Here we are concerned with both: repairable and non repairable systems. Apart from reliability, we are concerned also by several related measures as availability, maintainability, safety, mean times to failure, etc., since all of them are included into reliability evaluation of real systems.

For example, when the temporal evolution of a system is described by a stochastic process, say Z, with state space E, it is necessary in reliability to define a subset of E, say D, including all failed states of the system. Then the lifetime of such a system is defined by

$$T := \inf\{t \geq 0 : Z_t \in D\}.$$

The reliability is then defined by $R(t) := \mathbf{P}(T > t)$.

Of course, such a definition means that the system is binary, i.e., each state has to be an up or a down state. No degradation is considered. So, we consider the standard reliability model with catalectic failure modes (i.e., sudden and complete). For example, for a two component parallel system, which is in up state when at least one of the two components is in up state, one considers four states: $12, \bar{1}2, 1\bar{2}, \bar{1}\bar{2}$, where i means up state and $\bar{i}$, means down state of component $i = 1, 2$. The degradation states $\bar{1}2$, $1\bar{2}$ together with the perfect state 12 are considered as up states. Nevertheless, by different definitions of the set D, for the same system, one can cover several cases of degradation by the same formulae presented here.

Stochastic processes are used from the very beginning of reliability theory and practice. Markov processes, especially birth and death processes, Poisson processes, but also semi-Markov processes, especially the renewal processes, and some more general ones are used nowadays (Barbu and Limnios 2008; Barlow and Prochan 1975; Limnios and Oprişan 2001; Keilson 1979; Kovalenko et al. 1997; Esary et al. 1973; Gertsbakh 2000; Limnios and Nikulin 2000).

The use of such stochastic processes in reliability studies is usually limited to finite state space, (see, e.g., Barbu and Limnios 2008, Csenki 1995, Chryssaphinou et al. 2010, Janssen and Manca 2006, Lisnianski and Levitin 2003, Ouhbi and Limnios 2002, Ouhbi and Limnios 2003, Pérez-Ocón and Torres-Castro 2002). But in many real cases, on the one hand, the finite case, even the countable case, is not enough in order to describe and model the reliability of a real system. For example, in many cases, the state space can be $\{0, 1\}^{\mathbf{N}}$, in communication systems, or $\mathbf{R}_+$ in fatigue crack growth modeling (Chiquet et al. 2009; Devooght 1997). On the other hand, the Markov case is a restricted one since the lifetimes of many industrial components as the mechanical ones are not exponentially distributed but mostly they are Weibull, log-normal, etc. In maintenance the used distributions for durations are also not exponentially distributed, but mostly, log-normal, and also fixed duration time are considered. Nowadays, an increasing interest for semi-Markov modeling in reliability studies of real life systems is observed.

This is why we present a systematic modeling of reliability measures in the framework of semi-Markov processes in a general state space. Particular cases of this modeling are the discrete state space, the discrete time, and the Markov and renewal processes. The results presented here are new and generalize results in the finite state space case, (see Limnios and Oprişan 2001, Barbu and Limnios 2008, Lisnianski and Levitin 2003).

While the discrete-time case can be obtained from the continuous one, by considering counting measure for discrete time points, we consider that it is important to give separately this case since an increasing interest is observed in practice for the discrete case, (see, e.g., Barbu and Limnios 2008, Chryssaphinou et al. 2010, Lisnianski and Levitin 2003). The discrete-time model, on one hand, is much simpler to handle numerically than the continuous-time one. On the other hand, it can used to handle numerically continuous-time formulated problems. So, for practical reliability problems it is better to work in discrete-time.

This paper is organized as follows. Section 2 presents some definitions and notation of semi-Markov processes in general state space which are needed in the sequel. Section 3 presents the continuous time reliability modeling. The reliability, availability, failure rate, rocof and mean times are then defined into the semi-Markov setting and their explicit formulae are obtained via the Markov renewal equation. Section 4 presents the proofs of results given in Section 3. Section 5 is devoted to the discrete time semi-Markov processes with general state space toward reliability modeling. In the last section some general remarks for straightforward extensions of the present models are also presented.

2 The Semi-Markov Setting

In this section, we present the semi-Markov process framework which is needed in the sequel. Let us consider a regular semi-Markov process Z with state space the measurable space $(E, \mathcal{E})$, with countably generated σ-algebra, (see, e.g., Limnios and Oprişan 2001). Let $\mathbf{N} = \{0, 1, 2, ...\}$, and $\mathbf{R}_+ = [0, \infty)$, be the natural and nonnegative real numbers correspondingly.

Let $(J_n, S_n), n \geq 0$, be the (embedded) Markov Renewal Process (MRP) of Z, where $S_0 \leq S_1 \leq ...$ are the jump times and J_n are the visited states. This process satisfies the following Markov property

$$\mathbf{P}(J_{n+1} \in B, S_{n+1} - S_n \leq t \mid J_0, ..., J_n, S_0, ..., S_n) = \mathbf{P}(J_{n+1} \in B, S_{n+1} - S_n \leq t \mid J_n),$$

almost surely (a.s.), for any $n \in \mathbb{N}$, $t \in \mathbb{R}_+$, and $B \in \mathcal{E}$. The semi-Markov kernel Q, and the initial probability α are defined as follows:

$$Q(x, B, t) := \mathbf{P}(J_{n+1} \in B, S_{n+1} - S_n \leq t \mid J_n = x),$$

with $x \in E$, and $\alpha(B) := \mathbf{P}(J_0 \in B)$. We are using also the notation $Q(x, B, \Gamma)$ instead of $\int_\Gamma Q(x, B, ds)$, for Γ a measurable subset of $\mathbf{R}_+$. As usual, we denote by $\mathbf{P}_x(\cdot)$ the conditional probability $\mathbf{P}(\cdot \mid J_0 = x)$ and by $\mathbf{E}_x$ the corresponding expectation operator.

The counting process of jumps $N(t),\ t \geq 0$, defined by $N(t) = \sup\{n \geq 0 : S_n \leq t\}$, gives the number of jumps of the Markov renewal process in the time interval $(0, t]$. The semi-Markov process Z is defined by the relation

$$Z(t) = J_{N(t)}, \quad t \geq 0.$$

We suppose that Z is regular, which means that the number of jumps in a finite time interval is finite *a.s.* We suppose also that Z is cadlag, i.e., continuous on the right having left limits in any point of time $t > 0$.

The transition kernel of the embedded Markov chain (EMC) (J_n) is $P(x, B) := Q(x, B, \infty) = \lim_{t\to\infty} Q(x, B, t)$. Let us define the function $F_x(t) := Q(x, E, t)$ which is the distribution function of the holding (sojourn) time in state $x \in E$. The semi-Markov kernel has the following representation

$$Q(x, B, t) = \int_B F(x, y, t) P(x, dy), \quad B \in \mathcal{E}, \tag{1}$$

where $F(x, y, t) := \mathbf{P}(S_{n+1} - S_n \leq t \mid J_n = x, J_{n+1} = y)$. Define also the r.v. $X_{n+1} := S_{n+1} - S_n$, for $n \geq 0$, $X_0 = 0$, and the natural filtration $\mathcal{F}_n := \sigma(J_k, X_k; 0 \leq k \leq n)$, $n \geq 0$.

Let us consider a real-valued measurable function φ on $E \times \mathbf{R}_+$, and define its convolution by Q as follows

$$(Q * \varphi)(x, t) = \int_E \int_0^t Q(x, dy, ds)\varphi(y, t - s),$$

for $x \in E$, and $t \geq 0$. For two semi-Markov kernels, say Q_1 and Q_2, on $(E, \mathcal{E})$, one defines their convolution by

$$(Q_1 * Q_2)(x, B, t) = \int_E \int_0^t Q_1(x, dy, ds) Q_2(y, B, t - s),$$

where $x \in E$, $t \in \mathbf{R}_+$, $B \in \mathcal{E}$. The function $Q_1 * Q_2$ is also a semi-Markov kernel.

For a semi-Markov kernel Q on $(E, \mathcal{E})$, we set by induction

$$Q^{(1)} = Q, \quad Q^{(n+1)} = Q * Q^{(n)}, \quad n \geq 0, \quad \text{and} \quad Q^{(0)}(x, B, t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \mathbf{1}_B(x) & \text{if } t > 0, \end{cases}$$

where, for a set B, $\mathbf{1}_B$ denotes the indicator function of B, i.e., $\mathbf{1}_B(x) = 1$ for $x \in B$ and $\mathbf{1}_B(x) = 0$ for $x \notin B$.

It is clear that

$$Q^{(m+n)} = Q^{(m)} * Q^{(n)}.$$

Note that

$$Q^{(n)}(x, B, t) = \mathbf{P}(J_n \in B, S_n \leq t \mid J_0 = x).$$

The function ψ defined by

$$\psi(x, B, t) = \sum_{n\geq 0} Q^{(n)}(x, B, t),$$

is the *Markov Renewal function*.

For a kernel $K(x, B)$, a measure μ, and a point measurable function f, all defined on $(E, \mathcal{E})$, we define:

$$\mu K(B) := \int_E \mu(dx) K(x, B), \quad K f(x) := \int_E K(x, dy) f(y),$$

and

$$\mu K f = \int_{E \times E} \mu(dx) K(x, dy) f(y).$$

Let us now consider two real-valued functions $U(x, t)$ and $V(x, t)$ defined on $E \times \mathbf{R}_+$. The *Markov renewal equation* (MRE) is defined as follows

$$U(x, t) = V(x, t) + \int_E \int_0^t Q(x, dy, ds) U(y, t - s), \quad x \in E, \tag{2}$$

where U is an unknown function and V a given function.

From the Markov renewal theorem, given below, we have that Eq. 2 has a unique solution, and we also get the limit of this solution as $t \to \infty$.

Let us denote by $\mathbf{B}_1$ the set of all functions, say V, defined on $E \times \mathbf{R}$ which satisfy the following two properties:

- $V(x, t) = 0$ for $t \leq 0$ and any $x \in E$;
- V is uniformly bounded on E on every bounded subset of $\mathbf{R}_+$, i.e.,

$$\|V\|_{\infty,t} := \sup_{(x,s) \in E \times [0,t]} |V(x, s)| < +\infty,$$

for every $t > 0$. For example, function $\bar{F}_x(t) := 1 - F_x(t)$, belongs to space $\mathbf{B}_1$.

The MRE (2) has a solution, provided that function V belongs to $\mathbf{B}_1$, given by

$$U(x, t) = (\psi * V)(x, t) = \int_E \int_0^t \psi(x, dy, ds) V(y, t - s),$$

which also belongs to $\mathbf{B}_1$.

Let us here formulate the following assumptions:

C1: the stochastic kernel $P(x, B) := Q(x, B, \infty)$ induces an irreducible ergodic Markov chain with stationary distribution ν;

C2: the mean sojourn times are uniformly bounded, that is:

$$m(x) := \int_0^\infty \bar{F}_x(t) dt \leq C < +\infty,$$

and

$$m := \int_E \nu(dx) m(x) > 0;$$

C3: the distribution functions $F_x(t)$, $x \in E$, are *non arithmetic* (that is, not concentrated on a set $\{nb : n \in \mathbf{N}\}$, where $b > 0$, is a constant);

C4: the function $V(x, t)$ is direct Riemann integrable, (see, e.g., Feller 1966, Asmussen 1987), on $\mathbf{R}_+$, so

$$\int_E \nu(dx) \int_0^\infty |V(x, t)|\, dt < +\infty.$$

It is worth noticing that when $\mathbf{E}_x[X_1] < \infty$, then the function $\bar{F}_x$ is direct Riemann integrable. So, the first condition in C2 implies that $\bar{F}_x$ is direct Riemann integrable.

Theorem 2.1 *(Markov renewal theorem Shurenkov 1984) Under conditions C1-C4, Eq. 2 has a unique solution $U(x,t) = \psi * V(x,t)$, and the following result holds*

$$\lim_{t\to\infty} U(x,t) = \int_E \nu(dx) \int_0^\infty V(x,t)dt/m.$$

Let π be the stationary probability of an ergodic semi-Markov process Z. From the Markov renewal theorem we get $\pi(dx) = \nu(dx)m(x)/m$ (see, e.g., Limnios and Oprişan 2001).

In this paper, as it is stated above, we assume that the semi-Markov process Z is regular, that is $\mathbf{P}_x(N(t) < \infty) = 1$, for any $x \in E$ and any $t \geq 0$. A sufficient condition for a semi-Markov process to be regular is given in the following proposition.

Proposition 2.1 *If there exists a real number $a > 0$ such that $\|F\|_{\infty,a} < 1$, then the semi-Markov process Z is regular.*

We suppose that the assumption of Proposition 2.1 holds in the whole paper.

3 Reliability in Continuous Time Modeling

In the sequel we consider a stochastic system the time behaviour of which is described by a semi-Markov process Z as defined in the previous section. Consider now a measurable set U in $\mathcal{E}$ containing all up (working) states of the system. The set $E \setminus U$, denoted by D, contains all down (failed) states of the system. In order to avoid trivialities we suppose that $0 < \pi(U) < 1$, where π is the stationary probability distribution of Z. The transition from one state to another state, arising at jump times S_n of the semi-Markov process Z, means, physically speaking, the failure or the repair of at least one of the components of the system. The system is operational in U, but no service is delivered if it is in D. Nevertheless, in the case of repairable systems, a repair will return the system from D to U, otherwise, in the case of non repairable systems, it will remain in D forever (see, e.g., Limnios and Oprişan 2001). Let $\alpha(B)$, $B \in \mathcal{E}$, be the initial distribution of Z.

Reliability Let T be the hitting time of D, i.e.,

$$T := \inf\{t \geq 0 : Z_t \in D\}, \quad (\inf \emptyset = +\infty),$$

and the conditional reliability

$$R_x(t) := \mathbf{P}_x(T > t) = \mathbf{P}_x(Z_s \in U, \forall s \leq t), \quad x \in U.$$

Of course, $R_x(t) = 0$ for any $x \in D$ and $t \geq 0$. The (unconditional) reliability is defined by $R(t) := \mathbf{P}(T > t)$.

Let ψ_0 be the Markov renewal function, corresponding to the sub semi-Markov kernel Q_0, which is the restriction of the semi-Markov kernel Q on $U \times \mathcal{E}_U$, and $x \in E$, $B \in \mathcal{E}_U := \mathcal{E} \cap U$, $t \geq 0$.

Proposition 3.1 *The conditional reliability function $R_x(t)$ satisfies the Markov renewal equation (MRE):*

$$R_x(t) = \bar{F}_x(t) + \int_U \int_0^t Q(x, dy, ds) R_y(t-s), \quad x \in U. \tag{3}$$

Hence, the conditional reliability is given by the unique solution of the above equation, i.e.,

$$R_x(t) = (\psi_0 * \bar{F}_{\cdot}\mathbf{1}_U)(x, t),$$

where $\bar{F}_{\cdot}\mathbf{1}_U(x, t) = \mathbf{1}_U(x)\bar{F}_x(t)$.

And, finally, the (unconditional) reliability is

$$R(t) = \alpha(\psi_0 * \bar{F}_{\cdot}\mathbf{1}_U)(t) = \int_U \int_U \int_0^t \alpha(dx)\psi_0(x, dy, ds)\bar{F}_y(t-s). \tag{4}$$

The reliability function is defined for both kind of systems: non repairable (states U are transient) and repairable (the system is ergodic or assumptions C1–C2 hold). In the first case T denotes the absorption time into the set D and in the latter one it denotes the (first) hitting time of the set D.

Remark 3.1 Formula (4) generalizes the phase type distribution functions (see, e.g., Neuts 1981). In biomedical application the function $R(t)$ is called survival function.

Availability In fact, we have several types of availability. The most commonly used ones are the following.

1. *Point availability*. This is the availability of a system at time $t \geq 0$. Define the conditional point availability by

$$A_x(t) := \mathbf{P}_x(Z_t \in U), \quad x \in E.$$

Proposition 3.2 *The conditional availability function $A_x(t)$ satisfies the MRE:*

$$A_x(t) = \mathbf{1}_U(x)\bar{F}_x(t) + \int_E \int_0^t Q(x, dy, ds) A_y(t-s).$$

Hence, the conditional availability is given by the unique solution of the above equation, i.e.,

$$A_x(t) = (\psi * \bar{F}_{\cdot}\mathbf{1}_U)(x, t),$$

and the (unconditional) availability is

$$A(t) = \alpha(\psi * \bar{F}_{\cdot}\mathbf{1}_U)(t) = \int_E \int_U \int_0^t \alpha(dx)\psi(x, dy, ds)\bar{F}_y(t-s). \tag{5}$$

Since $\{Z_s \in U, 0 \leq s \leq t\} \subset \{Z_t \in U\}$, we obtain the important inequality $R(t) \leq A(t)$, for all $t \geq 0$. In case where all states U are transient, then formula (5) can be reduced to (4) and then we have $A(t) = R(t)$ for all $t \geq 0$.

2. *Steady-state availability*. This is the point wise availability under stationary process Z, that is

$$A_\infty := \mathbf{P}_\pi(Z_t \in U).$$

Of course, the above probability is independent of t.
Again from the Markov renewal theorem, applied to (5), it is easy to see that $\lim_{t\to\infty} A_x(t) = A_\infty$.

Proposition 3.3 *Under assumptions C1–C2, the steady-state availability is*

$$A_\infty = \frac{1}{m}\int_U \nu(dx)m(x) = \pi(U).$$

3. *Average availability*. The average availability over $(0, t]$ is defined by

$$\widetilde{A}(t) := \frac{1}{t}\int_0^t A(s)ds. \tag{6}$$

It is clear that under assumptions C1–C3 the system is ergodic, and then we have $\lim_{t\to\infty} \widetilde{A}(t) = A_\infty$.
4. *Interval availability*. This is the probability that system is in the set of up states U at time t and it will remain there during the time interval $[t, t+s]$, denoted by $A(t, s)$ $(t \geq 0, s \geq 0)$, that is

$$A(t, s) := \mathbf{P}(Z_u \in U, \forall u \in [t, t+s]).$$

This generalize the reliability and availability functions, since: $A(0, t) = R(t)$ and $A(t, 0) = A(t)$, (see, e.g., Aven and Jensen 1999). Denote by $A_x(t, s)$ the conditional interval availability given the event $\{Z_0 = x\}$.

Proposition 3.4 *The interval availability function $A_x(t, s)$ satisfies the MRE:*

$$A_x(t, s) = \mathbf{1}_U(x)\bar{F}_x(t+s) + \int_U \int_0^{t+s} Q(x, dy, du)A_y(t-u, s).$$

Hence, the interval availability is given by the unique solution of the above equation, and the (unconditional) interval availability is

$$A(t, s) = \int_E \int_U \int_0^{t+s} \alpha(dx)\psi(x, dy, du)\bar{F}_y(t+s-u). \tag{7}$$

And under the additional assumptions C1-C3, the limiting interval availability, for fixed $s > 0$, is given by

$$A_\infty(s) := \lim_{t\to\infty} A_x(t, s) = \frac{1}{m}\int_U \nu(dx)\int_0^\infty \bar{F}_y(t+s)dt.$$

The following additional assumptions will be used in the sequel:

A1: $\|F\|_{\infty,\Delta} = O(\Delta)$, for $\Delta \downarrow 0$;
A2: the semi-Markov kernel Q is absolutely continuous with respect to Lebesgue measure on $\mathbf{R}_+$, with Radon-Nikodym derivative q, that is $Q(x, B, dt) = q(x, B, t)dt$, for all $x \in E$, and $B \in \mathcal{E}$;

A3: the function $q(x, B, \cdot)$ is direct Riemann integrable, for any $x \in E$, and $B \in \mathcal{E}$ (see, e.g., Shurenkov 1984, Asmussen 1987, Korolyuk and Limnios 2005).
Let us also define the hazard rate function $\lambda(x, t)$ of the holding time distribution in state $x \in E$, $F_x(t)$, that is,

$$\lambda(x, t) = \frac{1}{\bar{F}_x(t)} \frac{d}{dt} F_x(t), \quad t \geq 0.$$

A4: The function $\lambda(x, t)$ exists for every $x \in E$, and belongs to the space $\mathbf{B}_1$.

Remark 3.2 It is worth noticing that assumption A2 implies that λ exists.

Mean times The mean times presented here play an important role in reliability practice.

MTTF: Mean Time To Failure. Let us define the conditional mean time to failure function $MTTF_x$ by

$$MTTF_x := \begin{cases} \mathbf{E}_x[T] & \text{if } x \in U \\ 0 & \text{if } x \in D, \end{cases}$$

and the (unconditional) mean time to failure by $MTTF := \mathbf{E}[T]$.

Proposition 3.5 *The $MTTF_x$ is a superharmonic function and satisfy the Poisson equation*

$$(I - P_0) MTTF_x = m(x), \quad x \in U$$

where P_0 is the restriction of P on $U \times \mathcal{E}_U$. Then we have

$$MTTF_x = G_0 m(x),$$

where G_0 is the potential operator of P_0, that is, $G_0 = \sum_{n \geq 0} (P_0)^n$, (see, e.g., Revuz 1975), and

$$MTTF = \alpha G_0 m. \tag{8}$$

MUT: Mean Up Time. This time concerns the mean up time under the condition that the underlying semi-Markov process is in steady-state. We obtain this mean time by calculating first the entry distribution to U set under stationary process assumption:

Proposition 3.6 *Under assumptions C1–C2, and A2, the entry distribution in U is*

$$\beta(B) = \lim_{t \to \infty} \mathbf{P}(Z_t \in B \mid Z_{t-} \in D, Z_t \in U) = \mathbf{P}_\nu(J_{n+1} \in B \mid J_n \in D, J_{n+1} \in U)$$
$$= \frac{\int_D \nu(dx) P(x, B)}{\int_D \nu(dx) P(x, U)}.$$

Consequently, we have the following result.

Proposition 3.7 *Under assumptions C1–C2, the MUT is:*

$$MUT = \frac{1}{(\nu 1_D P)(U)} \int_U (\nu 1_D P)(dy) G_0 m(y).$$

The detailed form of the above formula is as follows

$$MUT = \frac{1}{\int_D \nu(dx) P(x, U)} \int_D \int_U \int_U \nu(dx) P(x, dy) G_0(y, dz) m(z).$$

Remark 3.3 The corresponding MDT (Mean Down Time) is obtained by interchanging sets U and D.

Rate of occurrence of failures (rocof). The failure counting point process gives the number of times the semi-Markov process Z has visited set D in the time interval $(0, t]$, that is,

$$N_F(t) := \sum_{s \le t} \mathbf{1}_{\{Z_{t-} \in U,\ Z_t \in D\}}.$$

It is worth noticing that the above summation is meaningful since in the whole paper the process Z is supposed to be regular. The results here are generalizations of those obtained in Ouhbi and Limnios (2002) for the finite state space case. The proof here is new, since the proof given in the finite case does not apply here.

The intensity of the process $N_F(t), t \ge 0$, or the rate of occurrence of failure, $ro(t)$, is defined by

$$ro(t) := \frac{d}{dt} \mathbf{E} N_F(t).$$

Proposition 3.8 *Under assumptions A1, A2 and A4 we have*

$$ro(t) = \int_E \int_U \int_D \int_0^t \alpha(dz) \psi(z, dx, du) q(x, dy, t - y),$$

and, under the additional assumption A3, the asymptotic rate of occurrence is

$$ro := \lim_{t \to \infty} ro(t) = \int_U \nu(dx) P(x, D)/m.$$

Failure rate of the system We suppose here that the distribution functions $F_x, x \in E$, are absolutely continuous with densities f_x. This holds under assumption A2.

The standard failure rate definition can be adapted to the semi-Markov framework as follows.

$$\lambda(t) = \lim_{\Delta \downarrow 0} \frac{1}{\Delta} \mathbf{P}(Z_{t+\Delta} \in D \mid Z_s \in U, s \le t).$$

So, it can be written

$$\lambda(t) = \frac{\alpha_0(\psi_0 * f_{.} \mathbf{1}_U)(t)}{\alpha_0(\psi_0 * \bar{F}_{.} \mathbf{1}_U)(t)}.$$

Residual lifetime The residual lifetime at time $t \geq 0$ is the time left up to the failure for a component starting its life at time 0 and it is still alive at time t.

The distribution of the residual lifetime is defined by

$$F_t(s) = \mathbf{P}(T \leq t + s \mid T > t) = 1 - \frac{R(t+s)}{R(t)}.$$

The mean residual lifetime is

$$\mathbf{E}[T - t \mid T > t] = \int_t^\infty R(u) \frac{du}{R(t)}.$$

Hence, in the semi-Markov setting we have

$$F_t(s) = 1 - \frac{\alpha_0(\psi_0 * \bar{F})(t+s)}{\alpha_0(\psi_0 * \bar{F})(t)}.$$

Example Let us consider a semi-Markov process Z with state space $E = [-2, 2]$ and $\mathcal{E} = \mathcal{B}([-2, 2])$ (the Borel sets of $[-2, 2]$), and semi-Markov kernel Q given by

$$Q(x, B, t) = \frac{1}{4} \int_B (2 - |y|) dy \big(1 - e^{-t^{|x|+\delta}}\big),$$

where $\delta > 0$, $B \in \mathcal{B}([-2, 2])$, $x \in E$ and $t \geq 0$ (Bhattacharya and Majumdar 2007).

Consider now that $U = [0, 2]$ and $D = [-2, 0)$. Then we have

$$Q_0(x, B, t) = \frac{1}{4} \int_B (2 - y) dy \big(1 - e^{-t^{x+\delta}}\big),$$

where $x \in [0, 2]$, $B \in \mathcal{B}([0, 2])$ and $t \geq 0$.

The stationary probability ν of the EMC (J) is given by

$$\nu(B) = \int_B (2 - |x|) dx/4,$$

and the mean sojourn time $m(x)$ in state x is given by $m(x) = \Gamma(1 + \frac{1}{|x|+\delta})$, where $\Gamma(y) = \int_0^\infty e^{-t} t^{y-1} dt$, for $y > 0$. So the stationary probability π of the semi-Markov process is

$$\pi(B) = \frac{1}{4m} \int_B (2 - |x|) \Gamma\Big(1 + \frac{1}{|x| + \delta}\Big) dx,$$

with

$$m = \frac{1}{2} \int_0^2 (2 - x) \Gamma\Big(1 + \frac{1}{x + \delta}\Big) dx.$$

The steady-state availability is given by

$$A_\infty = \frac{1}{4m} \int_0^2 (2 - x) \Gamma\Big(1 + \frac{1}{x + \delta}\Big) dx.$$

The asymptotic rocof is:

$$ro = \frac{3}{64m}.$$

The other reliability measures can be obtained by numerical procedures.

4 Proofs

Proof of Proposition 2.1 The proof of this proposition is based on the following lemma. □

Lemma 4.1 *Under assumption of Proposition* 2.1*, we have* $Q^{(n)}(x, E, t) \leq c_t \beta^n$*, where* $0 < \beta < 1$*, for any* $x \in E$ *and* $t > 0$*, and* $c_t = e^t$*.*

Proof Let us define $\mathcal{J}_n = \sigma(J_0, ..., J_n)$, $n \in \mathbf{N}$, and from the assumption we can write $\|F\|_{\infty,a} = 1 - \varepsilon < 1$, for some $\varepsilon \in (0, 1)$.

We have, for $k \leq n$,

$$\begin{aligned}\mathbf{E}_x\big(e^{-X_n} \mid \mathcal{J}_n\big) &= \int_0^\infty e^{-x} F(J_{n-1}, J_n, dx) \\ &= \int_0^a e^{-x} F(J_{n-1}, J_n, dx) + \int_a^\infty e^{-x} F(J_{n-1}, J_n, dx) \\ &\leq e^{-a} + (1-\varepsilon)(1-e^{-a}) = 1 - \varepsilon(1-e^{-a}) =: \beta.\end{aligned}$$

where, obviously, $0 < \beta < 1$. The last inequality follows from the above inequality. And now we conclude as follows,

$$\begin{aligned}Q^{(n)}(x, E, t) &= \mathbf{P}_x(S_n \leq t) = \mathbf{P}_x\big(e^{-S_n} \geq e^{-t}\big) \leq e^t \mathbf{E}_x e^{-S_n} \\ &= e^t \mathbf{E}_x\Big[\mathbf{E}_x\big(e^{-S_n} \mid \mathcal{J}_n\big)\Big] = e^t \mathbf{E}_x\Big[\prod_{k=1}^n \mathbf{E}_x\big(e^{-X_k} \mid \mathcal{J}_n\big)\Big] \leq e^t \beta^n = c_t \beta^n.\end{aligned}$$

□

So, Lemma 4.1 implies that the semi-Markov process is normal, that is $\psi(x, E, t) < \infty$, for any $x \in E$ and any $t \geq 0$, which implies that it is regular (see, e.g., Limnios and Oprişan 2001).

Proof of Proposition 3.1 Let us define the process $\bar{Z}_t, t \geq 0$, by

$$\bar{Z}_t = \begin{cases} Z_t, & \text{if } t < T \\ \Delta, & \text{if } t \geq T, \end{cases}$$

where T is the hitting time of D for Z_t, as defined earlier, and Δ an extra absorbing state. Then the process $\bar{Z}_t$ is a semi-Markov process with semi-Markov kernel given by

$$\bar{Q}(x, B, t) = \begin{cases} Q(x, B, t), & \text{if } B \in \mathcal{E}_U \\ Q(x, D, t), & \text{if } B = \{\Delta\}, \end{cases}$$

and $x \in U$. It is obvious that the restriction of $\bar{Q}$ on $U \times \mathcal{E}_U$ is equal to Q_0. Let $(\bar{J}_n, \bar{S}_n)$ be the corresponding MRP of $\bar{Z}_t$.

Now, by a renewal argument, we get:

$$
\begin{aligned}
R_x(t) &= \mathbf{P}_x(T > t, \bar{S}_1 > t) + \mathbf{P}_x(T > t, \bar{S}_1 \le t) \\
&= \bar{F}_x(t) + \mathbf{E}_x[\mathbf{P}_x(T > t, \bar{S}_1 \le t \mid \bar{J}_1, \bar{S}_1)] \\
&= \bar{F}_x(t) + \int_U \int_0^t Q_0(x, dy, ds)\mathbf{P}_y(T > t - s) \\
&= \bar{F}_x(t) + \int_U \int_0^t Q_0(x, dy, ds)\, R_y(t - s).
\end{aligned}
$$

This is a MRE of type (2). Finally, as $\bar{F}_x(t)$ is a bounded function, and then it belongs to $\mathbf{B}_1$, we obtain the unique solution of the Markov renewal equation for reliability as stated in the proposition. □

Proof of Proposition 3.2 By the same renewal argument, as in the above reliability case, we obtain the renewal equation for availability. □

Proof of Proposition 3.3 Since $\bar{F}_x(t)$, $x \in E$, are direct Riemann integrables functions, from the Markov renewal theorem, we get

$$
\lim_{t\to\infty} \int_U \int_0^\infty \psi(x, dy, ds)\bar{F}_y(t - s) = \frac{1}{m} \int_U \nu(dy) \int_0^\infty \bar{F}_y(t)dt,
$$

and the conclusion follows. □

Proof of Proposition 3.4 By a renewal argument on the events $\{S_1 \le t\}$, $\{t < S_1 \le t + s\}$ and $\{S_1 > t + s\}$, we conclude the proof. □

Proof of Proposition 3.5 By integration of the two sides of the Markov renewal equation (3), and by noticing that $MTTF_x = \int_0^\infty R_x(t)dt$, we obtain

$$
\begin{aligned}
MTTF_x &= \int_0^\infty R_x(t)dt \\
&= \int_0^\infty \bar{F}_x(t)dt + \int_0^\infty \int_U \int_0^t Q(x, dy, ds)\, R_y(t - s)dt \\
&= m(x) + \int_U \int_0^\infty Q(x, dy, ds) \int_0^\infty R_y(t)dt \\
&= m(x) + \int_U P(x, dy) MTTF_y.
\end{aligned}
$$

From which we obtain the desired results. □

Proof of Proposition 3.6 The entry $\mathbf{P}_z$-probability distribution at time t in up states U, denoted by $\beta_t(z, B)$ can be obtained as follows.

$$\begin{aligned}\beta_t(z, B) &= \mathbf{P}_z(Z_t \in B \mid Z_{t-} \in D, Z_t \in U)\\ &= \frac{\mathbf{P}_z(Z_t \in B, Z_{t-} \in D)}{\mathbf{P}_z(Z_{t-} \in D, Z_t \in U)}\\ &= \frac{\lim_{\Delta\downarrow 0} \frac{1}{\Delta}\mathbf{P}_z(Z_t \in B, Z_{t-\Delta} \in D)}{\lim_{\Delta\downarrow 0} \frac{1}{\Delta}\mathbf{P}_z(Z_{t-\Delta} \in D, Z_t \in U)}\\ &= \frac{\int_D\int_B \lim_{\Delta\downarrow 0} \frac{1}{\Delta}\mathbf{P}_z(Z_t \in dy, Z_{t-\Delta} \in dx)}{\int_D\int_U \lim_{\Delta\downarrow 0} \frac{1}{\Delta}\mathbf{P}_z(Z_{t-\Delta} \in dy, Z_t \in dx)}\\ &= \frac{\int_D\int_B\int_0^t \psi(z, dx, du)q(x, dy, t-u)}{\int_D\int_U\int_0^t \psi(z, dx, du)q(x, dy, t-u)}\\ &= \frac{\int_D\int_0^t \psi(z, dx, du)q(x, B, t-u)}{\int_D\int_0^t \psi(z, dx, du)q(x, U, t-u)}.\end{aligned}$$

The fifth equality is due to Lemma 4.3 (see below).

Now, by the Markov renewal theorem, we obtain

$$\int_D\int_0^t \psi(z, dx, du)q(x, B, t-u) \longrightarrow \frac{1}{m}\int_D \nu(dx)P(x, B), \quad t \to \infty,$$

and the desired result follows. □

Proof of Proposition 3.7 Putting $\beta(dx)$ instead of $\alpha(dx)$ distribution in the formula (8) of MTTF, we obtain the desired result. □

Proof of Proposition 3.8 Let us denote by $\Delta N(t) = N(t+\Delta) - N(t)$, (see definition in Section 2), and $\Delta N_F(t) = N_F(t+\Delta) - N_F(t)$, for $t \geq 0$ and $\Delta > 0$. □

We first prove the following lemmas.

Lemma 4.2 *Under assumption A1, we have $Q^{(n)}(x, E, \Delta) = O(\Delta^n)$, as $\Delta \downarrow 0$, for any $x \in E$.*

Proof We will prove this lemma by induction over $n \geq 1$. For $n = 1$, we have

$$Q(x, E, \Delta) \leq \sup_{(x,s)\in E\times[0,\Delta]} Q(x, E, s) = \|F\|_{\infty,\Delta} = O(\Delta), \quad \Delta \downarrow 0.$$

Suppose that it is true for $n \geq 1$, and we prove it for $n+1$ as follows.

$$\begin{aligned}Q^{(n+1)}(x, E, \Delta) &= \int_E\int_0^\Delta Q(x, dy, ds)Q^{(n)}(y, E, \Delta - s)\\ &\leq Q(x, E, \Delta) \sup_{(z,u)\in E\times[0,\Delta]} Q^{(n)}(z, E, u)\\ &= O\big(\Delta^{n+1}\big).\end{aligned}$$

□

Lemma 4.3 *For disjoint sets A and B in $\mathcal{E}$, and under assumption A2, we have*

$$\lim_{\Delta \downarrow 0} \frac{1}{\Delta} \mathbf{P}_z(Z_t \in A, Z_{t+\Delta} \in B) = \int_A \int_0^t \psi(z, dx, ds) q(x, B, t-s).$$

Proof For $\Delta \downarrow 0$, and $x \in A$, $y \in B$, we can write

$$\begin{aligned}
\mathbf{P}_z(Z_t \in dx, Z_{t+\Delta} \in dy) &= \sum_{n \geq 0} \int_0^t \mathbf{P}_z(Z_t \in dx, S_{N(t)} \in ds, N(t) = n, Z_{t+\Delta} \in dy) \\
&= \sum_{n \geq 0} \int_0^t \mathbf{E}_z \big[\mathbf{P}_z(Z_t \in dx, S_{N(t)} \in ds, N(t) \\
&\qquad\qquad = n, Z_{t+\Delta} \in dy \mid J_n, S_n) \big] \\
&= \sum_{n \geq 0} \int_0^t Q^{(n)}(z, dx, ds) \mathbf{P}_z \\
&\quad \times (J_{n+1} \in dy, t - s < X_{n+1} \leq t - s + \Delta \mid J_n = x) \\
&= \int_0^t \psi(z, dx, ds) Q(x, dy, (t-s, t-s+\Delta]).
\end{aligned}$$

And now, dividing by Δ and taking the limit as $\Delta \downarrow 0$, the result follows by dominated convergence theorem. □

Let us define the backward recurrence time process U_t, $t \geq 0$, as follows $U_t = t$, if $S_1 > t$ and $U_t = t - S_{N(t)}$ for $S_1 < t$.

Lemma 4.4 *Under assumption A4, we have*

$$\mathbf{P}_z(S_{N(t)+1} - t \leq \Delta) = O(\Delta), \quad \Delta \downarrow 0.$$

Proof Let us consider a small $\Delta > 0$ and set $X_{N(t)+1} := S_{N(t)+1} - S_{N(t)}$. We have (see, also Limnios and Oprişan 2001, p. 72)

$$\begin{aligned}
&\mathbf{P}_z(S_{N(t)+1} - t \leq \Delta \mid Z_t = x, U_t = s) \\
&= \mathbf{P}_z(X_{N(t)+1} \leq s + \Delta \mid Z_t = x, U_t = s) \\
&= \frac{\bar{F}_x(s) - \bar{F}_x(s+\Delta)}{\bar{F}_x(s)} \\
&= \lambda(x, s)\Delta + O(\Delta) = O(\Delta).
\end{aligned}$$

The last equality follows from the fact that $\lambda(x, s) \in \mathbf{B}_1$. Now, from the above equality, using the following equality

$$\mathbf{P}_z(S_{N(t)+1} - t \leq \Delta) = \mathbf{E}_z[\mathbf{P}_z(S_{N(t)+1} - t \leq \Delta \mid Z_t, U_t)],$$

the proof follows. □

Lemma 4.5 *Under assumptions $A1$ and $A4$, for any integer $k \geq 2$ and any fixed time $t \geq 0$, we have, as $\Delta \downarrow 0$,*

1. $\mathbf{P}(\Delta N(t) \geq k) = O\big(\Delta^k\big)$;
2. $\mathbf{P}(\Delta N_F(t) = k) = O\big(\Delta^k\big)$.

Proof

1. Firstly, we have

$$\begin{aligned}
&\mathbf{P}(S_{N(t)+k} - S_{N(t)+1} \leq \Delta)\\
&= \int_E \int_E \mathbf{P}\big(J_{N(t)+k} \in dy, S_{N(t)+k} - S_{N(t)+1} \leq \Delta \mid J_{N(t)+1} = x\big)\mathbf{P}(J_{N(t)+1} \in dx)\\
&= \int_E \int_E Q^{(k-1)}(x, dy, \Delta)\mathbf{P}(J_{N(t)+1} \in dx)\\
&= O\big(\Delta^{k-1}\big) \int_E \mathbf{P}(J_{N(t)+1} \in dx) = O\big(\Delta^{k-1}\big),
\end{aligned}$$

where the last equality follows from Lemma 4.2.
Secondly, by 1., we obtain

$$\begin{aligned}
\mathbf{P}(\Delta N(t) \geq k) &= \mathbf{P}(S_{N(t)+k} - t \leq \Delta)\\
&\leq \mathbf{P}\big(S_{N(t)+1} - t \leq \Delta, S_{N(t)+k} - S_{N(t)+1} \leq \Delta\big)\\
&= \mathbf{E}\big[\mathbf{1}_{\{S_{N(t)+1} - t \leq \Delta\}}\mathbf{P}\big(S_{N(t)+k} - S_{N(t)+1} \leq \Delta \mid \mathcal{F}_{N(t)+1}\big)\big]\\
&\leq \mathbf{P}(S_{N(t)+1} - t \leq \Delta) O\big(\Delta^{k-1}\big).
\end{aligned}$$

Finally, from Lemma 4.4, we obtain obtain the desired result.
2. From $\{\Delta N_F(t) = k\} \subset \{\Delta N(t) \geq k\}$ the conclusion follows directly. □

Lemma 4.6 *Under assumptions $A1$ and $A4$, for any integer $k \geq 1$, and any fixed $t \geq 0$, we have*

$$\sum_{k \geq 2} k\mathbf{P}(\Delta N_F(t) = k) = O(\Delta^2), \quad \Delta \downarrow 0.$$

Proof Let us set $M(t) := \|F\|_{\infty,t}$ for $t \leq a$ (where $a > 0$, see Proposition 2.1).
From Lemma 4.5, we get

$$\begin{aligned}
\mathbf{P}(\Delta N_F(t) = k) &\leq \mathbf{P}(\Delta N(t) \geq k)\\
&= \mathbf{P}(S_{N(t)+k} - t \leq \Delta)\\
&\leq \mathbf{P}\big(S_{N(t)+1} - t \leq \Delta, S_{N(t)+k} - S_{N(t)+1} \leq \Delta\big)\\
&= \mathbf{E}\big[\mathbf{1}_{\{S_{N(t)+1} - t \leq \Delta\}}\mathbf{P}\big(S_{N(t)+k} - S_{N(t)+1} \leq \Delta \mid \mathcal{F}_{N(t)+1}\big)\big]\\
&= \mathbf{E}\big[\mathbf{1}_{\{S_{N(t)+1} - t \leq \Delta\}} Q^{(k-1)}(J_{N(t)+1}, E, \Delta)\big]\\
&\leq \mathbf{E}\big[\mathbf{1}_{\{S_{N(t)+1} - t \leq \Delta\}} (Q(J_{N(t)+1}, E, \Delta))^{k-1}\big]\\
&\leq \mathbf{P}(S_{N(t)+1} - t \leq \Delta)[M(t)]^{k-1}.
\end{aligned}$$

And now, from Lemma 4.4, assumption A1, and for $\Delta \downarrow 0$, we get

$$\begin{aligned}\sum_{k\geq 2} k\mathbf{P}(\Delta N_F(t) = k) &\leq O(\Delta)\sum_{k\geq 2} k[M(\Delta)]^{k-1} \\ &= O(\Delta)O(\Delta) = O(\Delta^2),\end{aligned}$$

which concludes the proof. □

Lemma 4.7 *Under assumptions* $A1$ *and* $A4$*, we have, for any fixed* $t \geq 0$*,*

$$\mathbf{P}_z(\Delta N_F(t) = 1) = \mathbf{P}_z(Z_t \in U, Z_{t+\Delta} \in D) + O(\Delta^2), \quad \Delta \downarrow 0.$$

Proof We have, for $\Delta \downarrow 0$:

$$\begin{aligned}\mathbf{P}_z(\Delta N_F(t) = 1) &= \int_U \mathbf{P}_z(\Delta N_F(t) = 1, Z_t \in dx) \\ &= \int_U \mathbf{P}_z(Z_t \in dx)[\mathbf{P}_z(\Delta N_F(t) = 1, \Delta N(t) = 1 \mid Z_t = x) \\ &\quad + \mathbf{P}_z(\Delta N_F(t) = 1, \Delta N(t) \geq 2 \mid Z_t = x)] \\ &= \int_U \mathbf{P}_z(Z_t \in dx)\mathbf{P}_z(\Delta N_F(t) = 1, \Delta N(t) = 1 \mid Z_t = x) + O(\Delta^2) \\ &= \int_U \int_D \mathbf{P}_z(Z_t \in dx)\mathbf{P}_z(Z_{t+\Delta} \in dy, \Delta N_F(t) = 1 \mid Z_t = x) + O(\Delta^2) \\ &= \int_U \int_D \mathbf{P}_z(Z_t \in dx)\mathbf{P}_z(Z_{t+\Delta} \in dy \mid Z_t = x) + O(\Delta^2).\end{aligned}$$

The above third equality is due to Lemma 4.5(1). Hence, the proof is achieved. □

Proof of Proposition 3.8 Let us write the $ro(z, t)$ as follows

$$\begin{aligned}ro(z, t) &= \lim_{\Delta\downarrow 0} \frac{1}{\Delta}\mathbf{E}_z[N_F(t + \Delta) - N_F(t)] = \lim_{\Delta\downarrow 0} \frac{1}{\Delta} \sum_{k\geq 1} k\mathbf{P}_z(\Delta N_F(t) = k) \\ &= \lim_{\Delta\downarrow 0} \frac{1}{\Delta}\mathbf{P}_z(\Delta N_F(t) = 1) + \lim_{\Delta\downarrow 0} \frac{1}{\Delta} \sum_{k\geq 2} k\mathbf{P}_z(\Delta N_F(t) = k).\end{aligned}$$

Now, by Lemma 4.6, the second term in the rhs of the last equality is 0. And then, from Lemmas 4.7 and 4.3, we obtain

$$ro(z, t) = \int_U \int_D \lim_{\Delta\downarrow 0} \frac{1}{\Delta}\mathbf{P}_z(Z_t \in dx, Z_{t+\Delta} \in dy) = \int_U \int_D \int_0^t \psi(z, dx, ds)q(x, dy, t - s).$$

Finally, taking into account the initial distribution of Z, the desired result for $ro(t)$ is obtained.

The asymptotic rocof is obtained by direct application of the Markov renewal theorem. □

5 Reliability in Discrete Time

Discrete-time semi-Markov setting Let us consider a Markov renewal process $(J_k, S_k, k \in \mathbf{N})$ in discrete time $k \in \mathbf{N}$, with state space the measurable space $(E, \mathcal{E})$ with countably generated σ-algebra. The semi-Markov kernel q is defined by

$$q(x, B, k) := \mathbf{P}(J_{n+1} \in B, S_{n+1} - S_n = k \mid J_n = x),$$

where $x \in E, B \in \mathcal{E}, k, n \in \mathbf{N}$. We will denote also $q(x, B, \Gamma) := \sum_{k \in \Gamma} q(x, B, k)$, where $\Gamma \subset \mathbf{N}$.

The process (J_n) is the embedded Markov chain of the MRP (J_n, S_n) with transition kernel $P(x, dy)$. The semi-Markov kernel q can be written as

$$q(x, dy, k) = P(x, dy) f_{xy}(k),$$

where $f_{xy}(k) := \mathbf{P}(S_{n+1} - S_n = k \mid J_n = x, J_{n+1} = y)$, the conditional distribution of the sojourn time.

Consider a measurable function $\varphi : E \times \mathbf{N} \to \mathbf{R}$, and define its convolution by q as follows

$$q * \varphi(x, k) = \sum_{\ell=0}^{k} \int_E q(x, dy, \ell)\varphi(y, k - \ell).$$

Define now the n-fold convolution of q by itself, as follows:

$$q^{(n)}(x, B, k) = \sum_{\ell=0}^{k} \int_E q(x, dy, \ell) q^{(n-1)}(y, B, k - \ell), \quad n \geq 2,$$

further $q^{(1)}(x, B, k) = q(x, B, k)$, and $q^{(0)}(x, B, k) = \mathbf{1}_B(x)\mathbf{1}_{\mathbf{N}}(k)$. It is worth noticing that $q^{(n)}(x, B, k) = 0$, for $n > k$.

Define also the Markov renewal function ψ by

$$\psi(x, B, k) := \sum_{n=0}^{k} q^{(n)}(x, B, k). \tag{9}$$

Let us consider two functions $u, v : E \times \mathbf{N} \to \mathbf{R}$. The Markov renewal equation is here

$$u(x, k) = v(x, k) + \sum_{\ell=0}^{k} \int_E q(x, dy, \ell) u(y, k - \ell). \tag{10}$$

For example, the Markov Renewal function (9) can be written in the following form which is a particular MRE

$$\psi(x, B, k) = q^{(0)}(x, B, k) + \sum_{\ell=0}^{k} \int_E q(x, dy, \ell)\psi(y, B, k - \ell).$$

Let us define the semi-Markov chain $Z_k, k \in \mathbf{N}$ associated to the MRP (J_n, S_n). Define first the counting process of jumps, $N(k) := \max\{n \geq 0 : S_n \leq k\}$, and the semi-Markov chain is defined by

$$Z_k = J_{N(k)}, \quad k \in \mathbf{N}.$$

The mean sojourn time in state $x \in E$ is $m(x) = \sum_{k \geq 0} \bar{F}_x(k)$ and $m = \int_E \nu(dx) m(x)$.

Theorem 5.1 (Markov Renewal Theorem (Shurenkov 1984)) *Under the following assumptions:*

DA1*:* *the Markov chain (J_n) is ergodic with stationary distribution $\nu(B)$, $B \in \mathcal{E}$;*
DA2*:* $0 < m < \infty$*;*
DA3*:* *The function $v(x, k)$ is measurable and*

$$\int_E \nu(dx) \sum_{k \geq 0} |v(x, k)| < \infty.$$

Then Eq. 10 has a unique solution given by $\psi * v(x, k)$, and

$$\lim_{k \to \infty} \psi * v(x, k) = \frac{1}{m} \int_E \nu(dx) \sum_{k \geq 0} v(x, k).$$

The transition function of the semi-Markov chain is defined by

$$P_k(x, B) := \mathbf{P}(Z_k \in B \mid Z_0 = x), \quad x \in E, B \in \mathcal{E}.$$

This function fulfills the following MRE

$$P_k(x, B) = \mathbf{1}_B(x) \bar{F}_x(k) + \sum_{\ell=0}^{k} \int_E q(x, dy, \ell) P_{k-\ell}(y, B).$$

From the Markov renewal theorem, under assumptions DA1–DA2, we get the stationary distribution π of Z, as a limit

$$\lim_{k \to \infty} P_k(x, B) = \frac{1}{m} \int_B \nu(dx) m(x) =: \pi(B).$$

Reliability modeling Let us consider a stochastic system whose temporal behavior is described by a semi-Markov chain Z with state space $(E, \mathcal{E})$, semi-Markov kernel q, and the initial probability α.

As in the continuous time case, the state space E is split into two parts: U containing the up states and D containing the down states.

Define the hitting time T of D, i.e.,

$$T := \inf\{k \geq 0 : Z_k \in D\}, \quad (\inf \emptyset = +\infty),$$

and the conditional reliability by

$$R_x(k) := \mathbf{P}_x(T > k) = \mathbf{P}_x(Z_\ell \in U, \forall \ell \leq k), \quad x \in U.$$

Of course, $R_x(k) = 0$ for any $x \in D$.

Let us define the Markov renewal function ψ_0,

$$\psi_0(x, B, k) := \sum_{n=0}^{k} q_0^{(n)}(x, B, k)$$

where q_0 is the restriction of the semi-Markov kernel q on $U \times U$, and $x \in E$, $B \in \mathcal{E}_U$, $k \in \mathbf{N}$.

Proposition 5.1 *The reliability function $R_x(k)$ satisfies the MRE:*

$$R_x(k) = \bar{F}_x(k) + \sum_{\ell=0}^{k} \int_U q(x, dy, \ell) R_y(k-\ell), \quad x \in U. \tag{11}$$

Hence, the reliability is given by the unique solution of the above equation by

$$R_x(k) = (\psi_0 * \bar{F}.\mathbf{1}_U)(x, k),$$

where $\bar{F}.\mathbf{1}_U(x, k) = \mathbf{1}_U(x)\bar{F}_x(k)$.

Finally the (unconditional) reliability is given by

$$R(k) = \sum_{\ell=0}^{k} \int_U \int_U \alpha(dx)\psi_0(x, dy, \ell)\bar{F}_y(k-\ell). \tag{12}$$

Remark 5.4 The above formula (12) is a generalization of the phase type distribution functions in the discrete-time case (see, e.g., Neuts 1981).

1. *Point availability*. This is the availability of a system at point time $k \in \mathbf{N}$. Define the conditional point availability

$$A_x(k) = \mathbf{P}_x(Z_k \in U), \quad x \in E.$$

Proposition 5.2 *The availability function $A_x(k)$ satisfies the MRE:*

$$A_x(k) = \mathbf{1}_U(x)\bar{F}_x(k) + \sum_{\ell=0}^{k} \int_E q(x, dy, \ell) A_y(k-\ell).$$

Hence, the availability is given by the unique solution of the above equation by

$$A_x(k) = (\psi * \bar{F}.\mathbf{1}_U)(x, k).$$

Finally the unconditional availability is given by

$$A(k) = \sum_{\ell=0}^{k} \int_E \int_U \alpha(dx)\psi(x, dy, \ell)\bar{F}_y(k-\ell). \tag{13}$$

2. *Steady-state availability*. This is the pointwise availability under the stationary process Z. That is, under assumptions DA1–DA2,

$$A_\infty = \mathbf{P}_\pi(Z_k \in U) = \int_U \nu(dx)m(x)/m.$$

Of course, the above probability is independent of k and it is given by the Markov renewal theorem as follows.
From the key renewal theorem applied to (13), it is easy to see that

$$\lim_{k\to\infty} A_x(k) = A_\infty.$$

3. *Average availability over* $[0, k]$. This availability is defined by

$$\widetilde{A}(k) := \frac{1}{k+1}\sum_{\ell=0}^{k} A(\ell).$$

4. *Interval availability* The interval availability in discrete time, $A(k, p)$, $k, p \in \mathbf{N}$, as in the continuous time case, is defined by

$$A(k, p) = \mathbf{P}(Z_\ell \in U, \forall \ell \in [k, k+p]).$$

Of course, we have: $R(k) = A(0, k)$ and $A(k) = A(k, 0)$.

Proposition 5.3 *The interval availability function $A_x(k, p)$ satisfies the MRE:*

$$A_x(k, p) = \mathbf{1}_U(x)\bar{F}_x(k+p) + \sum_{\ell=0}^{k+p}\int_U q(x, dy, \ell)A_y(k-\ell, p).$$

Hence, the interval availability is given by the unique solution of the above equation by

$$A_x(k, p) = (\psi * \bar{F}.\mathbf{1}_U)(x, k+p).$$

The unconditional interval availability is given by

$$A(k, p) = \alpha(\psi * \bar{F}.\mathbf{1}_U)(x, k+p). \tag{14}$$

The limiting interval availability, under assumptions DA1-DA2, for fixed $p > 0$, is given by

$$A_\infty(p) := \lim_{k\to\infty} A_x(k, p) = \frac{1}{m}\int_U \nu(dy)\sum_{k\geq 0}\bar{F}_y(k+p).$$

Mean times (i.e., MTTF, MUT, MDT) are given by the same formulae as in the continuous-time case.

6 Concluding Remarks

The reliability, availability, interval availability, mean times, rocof, etc., studied in this paper give the background and the tools for modeling and formulation of other measures. For example, the safety measures (related to the probability of catastrophic events) is obtained in the same way and the same formula (4) as the reliability except that the set D includes only the catastrophic failure states of the system. The maintainability, which is the probability of a system to be repaired up to time t, given that it is failed at time 0, is also obtained by the same formula as in the reliability case by only interchanging sets U and D, i.e.,

$$M(t) = 1 - \int_D\int_D\int_0^t \alpha(dx)\psi_1(x, dy, ds)\bar{F}_y(t-s),$$

where ψ_1 is the Markov renewal function, corresponding to the sub semi-Markov kernel Q_1 which is the restriction of Q on $D \times \mathcal{E}_D$.

Important other quantities connected to reliability are used in special fields as in software engineering, mechanical engineering, etc., where we can arrive at similar results.

Numerical algorithms have to be developed also in order to obtain numerical values for dependability measures using formulae given in this paper.

Statistical estimation of reliability can be based on the formulae presented here in order to obtain plug in estimators, (see, e.g., Limnios and Ouhbi 2003, Ouhbi and Limnios 2003, Ouhbi and Limnios 2002, Limnios and Ouhbi 2006).

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