Reliability of Semi-Markov Systems in Discrete Time: Modeling and Estimation

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Abstract

This chapter presents the reliability of discrete-time semi-Markov systems. After some basic definitions and notation, we obtain explicit forms for reliability indicators. We propose non-parametric estimators for reliability, availability, failure rate, mean hitting times and we study their asymptotic properties. Finaly, we give a three state example with detailled calculus and numerical evaluations.

Keywords: discrete-time semi-Markov systems, semi-Markov chains, reliability, availability, failure rate, mean hitting times, nonparametric estimation, asymptotic properties.

1 Introduction

In the last fifty years, a lot of work has been carried out in the field of probabilistic and statistical methods in reliability. We do not intend to provide here an overview of the field, but only to point out some bibliographical references which are close to the work presented in this chapter. More precisely, we are interested in discrete-time models for reliability and in models based on semi-Markov processes which extend the classical i.i.d. or markovian cases approach. The generality is important as we pass from a geometric distributed sojourn time in the Markov case, to a general distribution on the set of positive integers $\mathbb{N}$, as the discrete-time Weibull distribution.

It is worth noticing here that most of the mathematical models for reliability consider the time to be continuous. But there are real situations when systems have natural discrete lifetimes. We can cite here those systems which are working on demand, those working on cycles or those monitored only at certain discrete times (once a month, say). In such situations, the lifetimes are expressed in terms of the number of working periods, the number of working cycles or the

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number of months before failure. In other words, all these lifetimes are intrinsically discrete. But even in the continuous-time modeling case, we pass to the numerical calculus first by discetising the concerned model. A good overview of discrete probability distributions used in reliability theory can be found in [9].

Several authors have studied discrete-time models for reliability in a general i.i.d. setting (see [21, 9, 8]). The discrete time reliability modeling via homogeneous and non-homogeneous Markov chains can be found in [1, 22]. Statistical estimation and asymptotic properties for reliability metrics, using discrete-time homogeneous Markov chains, are presented in [23]. The continuous-time semi-Markov model in reliability can be found in [13, 18, 15].

As compared to the attention given to the continuous-time semi-Markov processes and related inference problems, the discrete-time semi-Markov processes are less studied. For an introduction to discrete-time renewal processes, see, for instance, [19]; an introduction to DTSMP can be found in [12], [16], [4]. The reliability of discrete-time semi Markov systems is investigated in [11, 5, 2, 3].

We give here a detailled modeling of reliability, availability, failure rate and mean up time with closed form solutions and statistical estimation based on a censured trajectory in the time interval $[0, M]$. The discrete time modeling, presented here, is more adapter for applications and is numerically easy to implement computer softwars to compute and estimate the abouve metrics.

The present chapter is structured as follows. In the first part, we define homogeneous discrete-time Markov renewal processes, homogeneous semi-Markov chains and we establish some basic notation. In Section 2, we consider a repairable discrete-time semi-Markov system and we obtain explicit forms for reliability measures: reliability, availability, failure rate and mean hitting times. Section 4 is devoted to nonparametric estimation. We first obtain estimators for the characteristics of o semi-Markov system. Then, we propose estimators for measures of the reliability and we present their asymptotic properties of the estimators. We end this chapter by a numerical application.

2 Semi-Markov Setting

In this section we define the discrete-time semi-Markov model, introduce the basic notation and definitions and present some probabilistic results on semi-Markov chains.

Consider a random system with finite state space $E = \{1, \dots, s\}$. We denote by $\mathcal{M}_E$ the set of non negative matrices on $E \times E$ and by $\mathcal{M}_E(\mathbb{N})$ the set of matrix-valued functions defined on $\mathbb{N}$, with values in $\mathcal{M}_E$. For $A \in \mathcal{M}_E(\mathbb{N})$, we write $A = (A(k);\ k \in \mathbb{N})$, where, for $k \in \mathbb{N}$ fixed, $A(k) = (A_{ij}(k);\ i, j \in E) \in \mathcal{M}_E$. Put $I_E \in \mathcal{M}_E$ for the identity matrix and $0_E \in \mathcal{M}_E$ for the null matrix.

We suppose that the evolution in time of the system is described by the following chains:

- the chain $J = (J_n)_{n \in \mathbb{N}}$ with state space E, where J_n is the system state at the n-th jump time;

- the chain $S = (S_n)_{n\in\mathbb{N}}$ with state space $\mathbb{N}$, where S_n is the n-th jump time. We suppose that $S_0 = 0$ and $0 < S_1 < S_2 < \ldots < S_n < S_{n+1} < \ldots$;
- the chain $X = (X_n)_{n\in\mathbb{N}^*}$ with state space $\mathbb{N}^*$, where X_n is the sojourn time in state J_{n-1} before the n-th jump. Thus, for all $n \in \mathbb{N}^*$, we have $X_n = S_n - S_{n-1}$.

One fundamental notion for semi-Markov systems is that of semi-Markov kernel in discrete time.

Definition 1 (Discrete-time semi-Markov kernel) *A matrix-valued function $\mathbf{q} \in \mathcal{M}_E(\mathbb{N})$ is said to be a discrete-time semi-Markov kernel if it satisfies the following three properties:*

1. $0 \le q_{ij}(k) \le 1,\ i,j \in E,\ k \in \mathbb{N}$,
2. $q_{ij}(0) = 0$ *and* $\sum_{k=0}^{\infty} q_{ij}(k) \le 1,\ i,j \in E$,
3. $\sum_{k=0}^{\infty}\sum_{j\in E} q_{ij}(k) = 1,\ i \in E.$

Definition 2 (Markov renewal chain) *The chain $(J,S) = (J_n, S_n)_{n\in\mathbb{N}}$ is said to be a Markov renewal chain (MRC) if for all $n \in \mathbb{N}$, for all $i,j \in E$ and for all $k \in \mathbb{N}$ it satisfies almost surely*

$$\begin{aligned} & \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n = k \mid J_0, \ldots, J_n; S_0, \ldots, S_n) \\ = \ & \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n = k \mid J_n). \end{aligned} \tag{1}$$

Moreover, if Equation (1) *is independent of n, (J,S) is said to be homogeneous and the discrete-time semi-Markov kernel $\mathbf{q}$ is defined by*

$$q_{ij}(k) := \mathbb{P}(J_{n+1} = j, X_{n+1} = k \mid J_n = i).$$

We also introduce the cumulative semi-Markov kernel as the matrix-valued function $\mathbf{Q} = (\mathbf{Q}(k);\ k \in \mathbb{N}) \in \mathcal{M}_E(\mathbb{N})$ defined by

$$Q_{ij}(k) := \mathbb{P}(J_{n+1} = j, X_{n+1} \le k \mid J_n = i) = \sum_{l=0}^{k} q_{ij}(l),\ i,j \in E,\ k \in \mathbb{N}. \tag{2}$$

Figure 1 gives a representation of the evolution of the system.

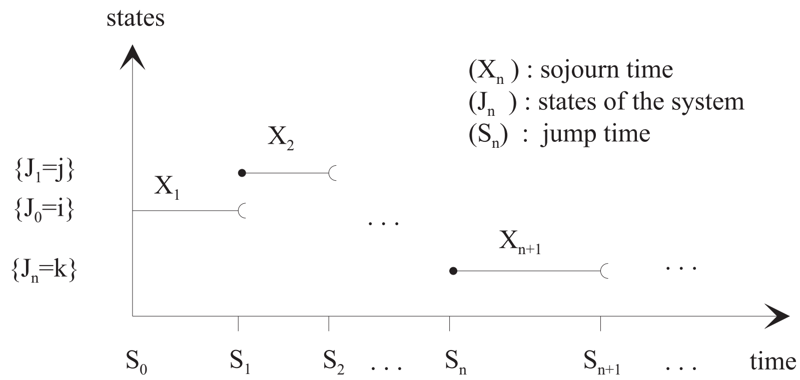

Figure 1: A typical sample path of a Markov renewal chain

Note that, for (J, S) a Markov renewal chain, we can easily see that $(J_n)_{n\in\mathbb{N}}$ is a Markov chain, called *the embedded Markov chain associated to the MRC* (J, S). We denote by $\mathbf{p} = (p_{ij})_{i,j\in E} \in \mathcal{M}_E$ the transition matrix of (J_n), defined by

$$p_{ij} = \mathbb{P}(J_{n+1} = j \mid J_n = i), \ i, j \in E, \ n \in \mathbb{N}.$$

We also assume that $p_{ii} = 0, \quad q_{ii}(k) = 0, k \in \mathbb{N}, i \in E$. We define now the sojourn time distributions in a given state and the conditional distributions depending on the next state to be visited.

Definition 3 (Conditional distributions of the sojourn times) *For all $i, j \in E$, let us define:*

1. $f_{ij}(\cdot)$, *the conditional distribution of* $X_{n+1}, \ n \in \mathbb{N}$:

$$f_{ij}(k) = \mathbb{P}(X_{n+1} = k \mid J_n = i, J_{n+1} = j), \ k \in \mathbb{N}. \tag{3}$$

2. $F_{ij}(\cdot)$, *the conditional cumulative distribution of* $X_{n+1}, \ n \in \mathbb{N}$:

$$F_{ij}(k) = \mathbb{P}(X_{n+1} \leq k \mid J_n = i, J_{n+1} = j) = \sum_{l=0}^{k} f_{ij}(l), \ k \in \mathbb{N}. \tag{4}$$

Obviously, for all $i, j \in E$ and for all $k \in \mathbb{N} \cup \{\infty\}$, we have

$$f_{ij}(k) = \begin{cases} q_{ij}(k)/p_{ij} & \text{if } p_{ij} \neq 0, \\ \mathbf{1}_{\{k=\infty\}} & \text{if } p_{ij} = 0. \end{cases} \tag{5}$$

Definition 4 (Sojourn times distributions in a given state) *For all $i \in E$, let us denote by:*

1. $h_i(\cdot)$, *the sojourn time distribution in state i:*

$$h_i(k) = \mathbb{P}(X_{n+1} = k \mid J_n = i) = \sum_{j\in E} q_{ij}(k), \ k \in \mathbb{N}^*.$$

2. $H_i(\cdot)$, *the sojourn time cumulative distribution function in state i:*

$$H_i(k) = \mathbb{P}(X_{n+1} \leq k \mid J_n = i) = \sum_{l=1}^{k} h_i(l), \ k \in \mathbb{N}^*.$$

Let us also denote by m_i the mean sojourn time in a state $i \in E$,

$$m_j := \mathbb{E}(S_1 \mid J_0 = j) = \sum_{n\geq 0} (1 - H_j(n)).$$

For G the cumulative distribution function of a certain r.v. X, we denote *the survival function* by $\overline{G}(n) := 1 - G(n) = \mathbb{P}(X > n)$, $n \in \mathbb{N}$. Thus, for all states $i, j \in E$ we put $\overline{F}_{ij}$ and $\overline{H}_i$ for the corresponding survival functions.

The operation which will be commonly used when working on the space $\mathcal{M}_E(\mathbb{N})$ of matrix-valued functions will be the discrete-time matrix convolution product. In the sequel we recall its definition, we see that there exists a neutral element, we define recursively the $n-$fold convolution and we introduce the notion of the inverse in the convolution sense.

Definition 5 (Discrete-time matrix convolution product) *Let* $A, B \in \mathcal{M}_E(\mathbb{N})$ *be two matrix-valued functions. The matrix convolution product* $A * B$ *is a matrix-valued function* $C \in \mathcal{M}_E(\mathbb{N})$ *defined by*

$$C_{ij}(k) := \sum_{r \in E} \sum_{l=0}^{k} A_{ir}(k-l)\, B_{rj}(l), \quad i, j \in E, \quad k \in \mathbb{N}.$$

The following result concerns the existence of the neutral element for the matrix convolution product in discrete time.

Lemma 1 *Let* $\delta I = (d_{ij}(k);\ i, j \in E) \in \mathcal{M}_E(\mathbb{N})$ *be the matrix-valued function defined by*

$$d_{ij}(k) := \begin{cases} 1 & \text{if } i = j \text{ and } k = 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Then, δI *satisfies*

$$\delta I * A = A * \delta I = A, \quad A \in \mathcal{M}_E(\mathbb{N}),$$

i.e., δI *is the neutral element for the discrete-time matrix convolution product.*

The power in the sense of convolution is defined straightforward, using Definition 5.

Definition 6 (Discrete-time $n-$**fold convolution)** *Let* $A \in \mathcal{M}_E(\mathbb{N})$ *be a matrix-valued function and* $n \in \mathbb{N}$*. The* $n-$*fold convolution* $A^{(n)}$ *is a matrix-valued function in* $\mathcal{M}_E(\mathbb{N})$ *defined recursively by:*

$$A_{ij}^{(0)}(k) := \begin{cases} 1 & \text{if} \quad k = 0 \quad \text{and} \quad i = j, \\ 0 & \text{elsewhere}, \end{cases}$$

$$A_{ij}^{(1)}(k) := A_{ij}(k)$$

and

$$A_{ij}^{(n)}(k) := \sum_{r \in E} \sum_{l=0}^{k} A_{ir}(l)\, A_{rj}^{(n-1)}(k-l), \quad n \geq 2, k \in \mathbb{N}.$$

For a MRC (J, S), the $n-$fold convolution of the semi-Markov kernel has the property expressed in the following result.

Lemma 2 *Let* $(J, S) = (J_n, S_n)_{n \in \mathbb{N}}$ *be a Markov renewal chain and* $\mathbf{q} = (q_{ij};\ i, j \in E) \in \mathcal{M}_E(\mathbb{N})$ *be its associated semi-Markov kernel. Then, for all* $n,\ k \in \mathbb{N}$ *such that* $n \geq k+1$ *we have* $\mathbf{q}^{(n)}(k) = 0$.

This property of the discrete-time semi-Markov kernel convolution is essential for the simplicity and the numerical exactitude of the results obtained in discrete time. We need to stress the fact that this property is intrinsic to the work in discrete time and it is not valid any more for a continuous-time Markov renewal process.

Definition 7 (Left inverse in the convolution sense) *Let* $A \in \mathcal{M}_E(\mathbb{N})$ *be a matrix-valued function. If there exists a* $B \in \mathcal{M}_E(\mathbb{N})$ *such that* $B * A = \delta I$*, then* B *is called the left inverse of* A *in the convolution sense and it is denoted by* $A^{(-1)}$.

It can be shown that given a matrix-valued function $A \in \mathcal{M}_E(\mathbb{N})$, if $\det A(0) \neq 0$, then the left inverse B of A exists and is unique (see [4] for the proof).

Let us now introduce the notion of semi-Markov chain, strictly related with that of Markov renewal chain.

Definition 8 (Semi-Markov chain) *Let (J, S) be a Markov renewal chain. The chain $Z = (Z_k)_{k\in\mathbb{N}}$ is said to be a semi-Markov chain associated to the MRC (J,S), if*

$$Z_k := J_{N(k)}, k \in \mathbb{N},$$

where

$$N(k) := \max\{n \in \mathbb{N} \mid S_n \leq k\} \tag{6}$$

is the discrete-time counting process of the number of jumps in $[1, k] \subset \mathbb{N}$. Thus, Z_k gives the system state at time k. We have also $J_n = Z_{S_n}, n \in \mathbb{N}$.

Let the row vector $\alpha = (\alpha_1, \ldots, \alpha_s)$ denote *the initial distribution* of the semi-Markov chain $Z = (Z_k)_{k\in\mathbb{N}}$, where $\alpha_i := \mathbb{P}(Z_0 = i) = \mathbb{P}(J_0 = i),\ i \in E$.

Definition 9 *The transition function of the semi-Markov chain Z is the matrix-valued function $\mathbf{P} \in \mathcal{M}_E(\mathbb{N})$ defined by*

$$P_{ij}(k) := \mathbb{P}(Z_k = j \mid Z_0 = i),\ i, j \in E,\ k \in \mathbb{N}.$$

The following result consists in a recursive formula for computing the transition function $\mathbf{P}$ of the semi-Markov chain Z.

Proposition 1 *For all $i, j \in E$ and for all $k \in \mathbb{N}$, we have*

$$P_{ij}(k) = \mathbf{1}_{\{i=j\}}(k)\,[1 - H_i(k)] + \sum_{r\in E}\sum_{l=0}^{k} q_{ir}(l)P_{rj}(k - l), \tag{7}$$

where

$$\mathbf{1}_{\{i=j\}}(k) := \begin{cases} 1 & \text{if } i = j \text{ and } k \geq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Let us define for all $k \in \mathbb{N}$:

- $I(k) := (\mathbf{1}_{\{i=j\}}(k);\ i, j \in E)$, $I := (I(k);\ k \in \mathbb{N})$;
- $\mathbf{H}(k) := diag(H_i(k);\ i \in E)$, $\mathbf{H} := (\mathbf{H}(k);\ k \in \mathbb{N})$.

In matrix-valued function notation, Equation (7) becomes

$$\mathbf{P} = I - \mathbf{H} + \mathbf{q} * \mathbf{P}. \tag{8}$$

Equation (8) is an example of what is called discrete-time Markov renewal equation. We know that the solution of this equation exists, is unique (see [4]) and has the following form

$$\mathbf{P}(k) = (\delta I - \mathbf{q})^{(-1)} * (I - \mathbf{H})(k) = (\delta I - \mathbf{q})^{(-1)} * (I - diag(\mathbf{Q}\ \cdot \mathbf{1}))(k),\ k \in \mathbb{N}. \tag{9}$$

3 Reliability Modeling

In this Section we consider a reparable discrete-time semi-Markov system and we obtain closed form solutions for reliability measures: reliability, availability, failure rate, mean time to failure, mean time to repair.

3.1 State Space Split

Consider a system (or a component) $\mathcal{S}$ whose possible states during its evolution in time are $E = \{1, \ldots, s\}$. Denote by $U = \{1, \ldots, s_1\}$ the subset of working states of the system (the up-states) and by $D = \{s_1 + 1, \ldots, s\}$ the subset of failure states (the down-states), with $0 < s_1 < s$ (obviously, $E = U \cup D$ and $U \cap D = \emptyset$, $U \neq \emptyset$, $D \neq \emptyset$). One can think the states of U as different operating modes or performance levels of the system, whereas the states of D can be seen as failures of the systems with different modes. According to the partition of the state space in up-states and down-states, we will partition the vectors, matrices or matrix functions we are working with.

Firstly, for α, $\mathbf{p}$, $\mathbf{q}(k)$, $\mathbf{f}(k)$, $\mathbf{F}(k)$, $\mathbf{H}(k)$, $\mathbf{Q}(k)$, we consider the natural matrix partition corresponding to the state space partition U and D. For example, we have

$$\mathbf{p} = \begin{array}{c} \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} U & D \end{array} \\ \left[\begin{array}{cc} \mathbf{p}_{11} & \mathbf{p}_{12} \\ \mathbf{p}_{21} & \mathbf{p}_{22} \end{array}\right] \end{array} \begin{array}{c} U \\ D \end{array}, \quad \mathbf{q}(k) = \begin{array}{c} \begin{array}{cc} U & D \end{array} \\ \left[\begin{array}{cc} \mathbf{q}_{11}(k) & \mathbf{q}_{12}(k) \\ \mathbf{q}_{21}(k) & \mathbf{q}_{22}(k) \end{array}\right] \end{array} \begin{array}{c} U \\ D \end{array},$$

$$\mathbf{H}(k) = \begin{array}{c} \begin{array}{cc} U & D \end{array} \\ \left[\begin{array}{cc} \mathbf{H}_{11}(k) & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{22}(k) \end{array}\right] \end{array} \begin{array}{c} U \\ D \end{array} = \begin{array}{c} \begin{array}{cc} U & D \end{array} \\ \left[\begin{array}{cc} diag(H_i(k))_{i \in U} & \mathbf{0} \\ \mathbf{0} & diag(H_i(k))_{i \in D} \end{array}\right] \end{array} \begin{array}{c} U \\ D \end{array}.$$

Secondly, for $\mathbf{P}(k)$ we consider the restrictions to $U \times U$ and $D \times D$ induced by the corresponding restrictions of the semi-Markov kernel $\mathbf{q}(k)$. To be more specific, using the partition given above for the kernel $\mathbf{q}$, we note:

- $\mathbf{P}_{11}(k) := (\delta I - \mathbf{q}_{11})^{(-1)} * (I - diag(\mathbf{Q} \cdot \mathbf{1})_{11})(k)$,
- $\mathbf{P}_{22}(k) := (\delta I - \mathbf{q}_{22})^{(-1)} * (I - diag(\mathbf{Q} \cdot \mathbf{1})_{22})(k)$.

The reasons fort taking this partition for $\mathbf{P}(k)$ can be found in [6].

For $m, n \in \mathbb{N}^*$ such that $m > n$, let $\mathbf{1}_{m,n}$ denote the $m-$dimensional column vector whose n first elements are 1 and last $m - n$ elements are 0; for $m \in \mathbb{N}^*$, let $\mathbf{1}_m$ denote the m-column vector whose elements are all 1, that is, $\mathbf{1}_m = \mathbf{1}_{m,m}$.

3.2 Reliability

Consider a system $\mathcal{S}$ starting to function at time $k = 0$ and let T_D denote the first passage time in subset D, called *the lifetime of the system*, i.e.,

$$T_D := \inf\{n \in \mathbb{N}; \quad Z_n \in D\} \text{ and } \inf \emptyset := \infty.$$

The reliability of a discrete-time semi-Markov system $\mathcal{S}$ at time $k \in \mathbb{N}$, that is the probability that the system has functioned without failure in the period $[0, k]$ is

$$R(k) := \mathbb{P}(T_D > k) = \mathbb{P}(Z_n \in U, n = 0, \ldots, k).$$

The following result gives the reliability of the system in terms of the basic quantities of the semi-Markov chain.

Proposition 2 *The reliability of a discrete-time semi-Markov system at time $k \in \mathbb{N}$ is given by*

$$R(k) = \alpha_1 \, \mathbf{P}_{11}(k) \, \mathbf{1}_{s_1} = \alpha_1 (\delta I - \mathbf{q}_{11})^{(-1)} * (I - diag(\mathbf{Q} \; \cdot \mathbf{1})_{11})(k) \mathbf{1}_{s_1}. \tag{10}$$

3.3 Availability

The pointwise (or instantaneous) availability of a system $\mathcal{S}$ at time $k \in \mathbb{N}$ is the probability that the system is operational at time k (independently of the fact that the system has failed or not in $[0, k)$).

So, the pointwise availability of a semi-Markov system at time $k \in \mathbb{N}$ is

$$A(k) = \mathbb{P}(Z_k \in U) = \sum_{i \in E} \alpha_i A_i(k),$$

where we have denoted by $A_i(k)$ the system's availability at time $k \in \mathbb{N}$, given that it starts in state $i \in E$,

$$A_i(k) = P(Z_k \in U \mid Z_0 = i).$$

The following results gives an explicit form of the availability af discrete-time semi-Markov system.

Proposition 3 *The pointwise availability of a discrete-time semi-Markov system at time $k \in \mathbb{N}$ is given by*

$$A(k) = \alpha \, \mathbf{P}(k) \, \mathbf{1}_{s,s_1} = \alpha \, (\delta I - \mathbf{q})^{(-1)} * (I - \mathbf{diag}(\mathbf{Q} \; \cdot \mathbf{1}))(k) \mathbf{1}_{s,s_1}. \tag{11}$$

3.4 Failure Rate

We consider here the classical failure rate, introduced by Barlow, Marshall and Prochan (1963). We call it *the BMP-failure rate* and denote it by $\lambda(k), k \in \mathbb{N}$.

Let $\mathcal{S}$ be a system starting to function at time $k = 0$. The BMP-failure rate at time $k \in \mathbb{N}$ is the conditional probability that the failure of the system occurs at time k, given that the system has worked until time $k - 1$.

For a discrete-time semi-Markov system, the failure rate at time $k \geq 1$ has the expression

$$\begin{aligned} \lambda(k) &:= \mathbb{P}(T_D = k \mid T_D \geq k) \\ &= \begin{cases} 1 - \frac{R(k)}{R(k-1)}, & R(k-1) \neq 0 \\ 0, & otherwise \end{cases} \\ &= \begin{cases} 1 - \frac{\alpha_1 \, \mathbf{P}_{11}(k) \, \mathbf{1}_{s_1}}{\alpha_1 \, \mathbf{P}_{11}(k-1) \, \mathbf{1}_{s_1}}, & R(k-1) \neq 0, \\ 0, & otherwise. \end{cases} \end{aligned} \tag{12}$$

The failure rate at time $k = 0$ is defined by $\lambda(0) := 1 - R(0)$.

It is worth noticing that the failure rate $\lambda(k)$ in discrete-time case is a probability function and not a general positive function as in the continuous-time case.

3.5 Mean Hitting Times

There are various mean times which are interesting for the reliability analysis of a system. We will be concerned here only with the mean time to failure and mean time to repair.

We suppose that $\alpha_2 = 0$, i.e. the system starts in a working state. The *Mean Time To Failure* (MTTF) is defined as the mean lifetime, i.e. the expectation of the hitting time to down set D,

$$MTTF = \mathbb{E}[T_D].$$

Symmetrically, consider now that $\alpha_1 = 0$, i.e. the system fails at the time $t = 0$. The *Mean Time To Repair* (MTTR) is defined as the mean of the repair duration, i.e. the expectation of the hitting time to up-set U,

$$MTTR = \mathbb{E}[T_U].$$

The following result gives expressions for the MTTF and MTTR of a discrete-time semi-Markov system.

Proposition 4 *If the matrices $I - \mathbf{p}_{11}$ and $I - \mathbf{p}_{22}$ are nonsingular, then*

$$MTTF = \alpha_1 (I - \mathbf{p}_{11})^{-1} \mathbf{m}_1, \tag{13}$$

$$MTTR = \alpha_2 (I - \mathbf{p}_{22})^{-1} \mathbf{m}_2, \tag{14}$$

where $\mathbf{m} = (\mathbf{m}_1 \quad \mathbf{m}_2)^\top$ is the partition of the mean sojourn times vector corresponding to the partition of state space E in up-states U and down-states D. If the matrices are singular, we put $MTTF = \infty$ or $MTTR = \infty$.

Remark 1 *One can prove that, if there exists an $i, 1 \leq i \leq s_1$, such that $\sum_{j=1}^{s_1} p_{ij} < 1$, then $I - \mathbf{p}_{11}$ is nonsingular. Symmetrically, if there exists an $l, s_1 + 1 \leq l \leq s$, such that $\sum_{j=s_1+1}^{s} p_{lj} < 1$, then $I - \mathbf{p}_{22}$ is nonsingular.*

So, under these conditions, the hypotheses of the above Proposition are fulfilled and we have the expressions of MTTF and MTTR given in Equations (13), respectively (14).

4 Reliability Estimation

The objective of this chapter is to provide estimators for reliability indicators of a system and to present their asymptotic properties. In order to achieve this purpose, we firstly show how estimators of the basic quantities of a discrete-time semi-Markov system are obtained.

4.1 Semi-Markov Estimation

Let us consider a sample path of a Markov renewal chain $(J_n, S_n)_{n\in\mathbb{N}}$, censored at fixed arbitrary time $M \in \mathbb{N}^*$,

$$\mathcal{H}(M) := (J_0, X_1, \ldots, J_{N(M)-1}, X_{N(M)}, J_{N(M)}, u_M),$$

where $N(M)$ is the discrete-time counting process of the number of jumps in $[1, M]$ (see Equation (6)) and $u_M := M - S_{N(M)}$ is the censored sojourn time in the last visited state $J_{N(M)}$.

Starting from the sample path $\mathcal{H}(M)$, we will propose empirical estimators for the quantities of interest. Let us firstly define the number of visits to a certain state, the number of transitions between two states and so on.

Definition 10 *For all states $i, j \in E$ and positive integer $k \leq M$, define:*

1. $N_i(M) := \sum_{n=0}^{N(M)-1} \mathbf{1}_{\{J_n=i\}}$ *- the number of visits to state i, up to time M;*
2. $N_{ij}(M) := \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_{n-1}=i, J_n=j\}}$ *- the number of transitions from i to j, up to time M;*
3. $N_{ij}(k, M) := \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_{n-1}=i, J_n=j, X_n=k\}}$ *- the number of transitions from i to j, up to time M, with sojourn time in state i equal to $k, 1 \leq k \leq M$.*

For a sample path of length M of a semi-Markov chain, for any states $i, j \in E$ and positive integer $k \in \mathbb{N}, k \leq M$, we define the empirical estimators of the transition matrix of the embedded Markov chain p_{ij}, of the conditional distributions of the sojourn times $f_{ij}(k)$ and of the discrete-time semi-Markov kernel $q_{ij}(k)$ by:

$$\widehat{p}_{ij}(M) := N_{ij}(M)/N_i(M), \tag{15}$$

$$\widehat{f}_{ij}(k, M) := N_{ij}(k, M)/N_{ij}(M), \tag{16}$$

$$\widehat{q}_{ij}(k, M) := N_{ij}(k, M)/N_i(M). \tag{17}$$

Note that the proposed estimators are natural estimators. For instance, the probability p_{ij} that the system goes from state i to state j is estimated by the number of transitions from i to j, up to time M, devised by the number of visits to state i, up to time $M - 1$. As it will be proved in the following section, the empirical estimators proposed in formulas (15) - (17) have good asymptotic properties. Moreover, they are in fact approached maximum likelihood estimators (Theorem 1). In order to see this, consider the likelihood function corresponding to the history $\mathcal{H}(M)$

$$L(M) = \prod_{k=1}^{N(M)} p_{J_{k-1}J_k} f_{J_{k-1}J_k}(X_k) \overline{H}_{J_{N(M)}}(u_M),$$

where $\overline{H_i}(\cdot)$ is the survival function in state i defined by

$$\overline{H_i}(n) := \mathbb{P}(X_1 > n \mid J_0 = i) = 1 - \sum_{j\in E} \sum_{k=1}^{n} q_{ij}(k),\ n \in \mathbb{N}^*.$$

We have the following result concerning the asymptotic behavior of u_M (see [6] for the proof).

Lemma 3 *For a semi-Markov chain $(Z_n)_{n\in\mathbb{N}}$ we have*

$$u_M/M \xrightarrow[M\to\infty]{a.s.} 0, \tag{18}$$

where $u_M = M - S_{N(M)}$.

Let us consider the approached likelihood function

$$L_1(M) = \prod_{k=1}^{N(M)} p_{J_{k-1}J_k} f_{J_{k-1}J_k}(X_k), \tag{19}$$

obtained by neglecting the last term in the expression of $L(M)$. Using Lemma 3, we see that the maximum likelihood function $L(M)$ and the approached maximum likelihood function $L_1(M)$ are asymptotically equivalent, as M tends to infinity. Consequently, the estimators obtained by estimating $L(M)$ or $L_1(M)$ are asymptotically equivalent, as M tends to infinity.

The following result shows that $\widehat{p}_{ij}(M)$, $\widehat{f}_{ij}(k, M)$ et $\widehat{q}_{ij}(k, M)$ defined by expressions (15) - (17) are obtained in fact by maximizing $L_1(M)$ (a proof can be found in [2])

Theorem 1 *For a sample path of a semi-Markov chain $(Z_n)_{n\in\mathbb{N}}$, of arbitrary fixed length $M \in \mathbb{N}$, the empirical estimators of the transition matrix of the embedded Markov chain $(J_n)_{n\in\mathbb{N}}$, of the conditional distributions of the sojourn times and of the discrete-time semi-Markov kernel, proposed in Equations (15) - (17), are approached nonparametric maximum likelihood estimators, i.e., they maximise the approached likelihood function L_1, given by Equation (19).*

As any quantity of interest of a semi-Markov system can be written in terms of the semi-Markov kernel, we can now use the kernel estimator (17) in order to obtain plug-in estimators for any functional of the kernel. So, the cumulative semi-Markov kernel $\mathbf{Q} = (\mathbf{Q}(k);\ k \in \mathbb{N})$ defined in (2) has the estimator

$$\widehat{\mathbf{Q}}(k, M) \quad := \quad \sum_{l=1}^{k} \widehat{\mathbf{q}}(l, M), \tag{20}$$

where $\widehat{\mathbf{q}}^{(n)}(k, M)$ is the n-fold convolution of $\widehat{\mathbf{q}}(k, M)$ (see Definition 6).

Similarly, using the expression of the transition function of the semi-Markov chain Z given in Equation (9), we get its estimator

$$\widehat{\mathbf{P}}(k, M) \quad := \quad \Big(\delta I - \widehat{\mathbf{q}}\Big)^{(-1)}(\cdot, M) * \Big(I - diag(\widehat{\mathbf{Q}}(\cdot, M) \cdot \mathbf{1})\Big)(k). \tag{21}$$

Proofs of the consistency and of the asymptotic normality of the estimators defined up to now can be found in [5, 2, 6].

We are able now to construct estimators of the reliability indicators of a semi-Markov system and to present their asymptotic properties.

4.2 Reliability Estimation

The expression of the reliability given in (10), together with the estimators of the semi-Markov transition function and of the cumulative semi-Markov kernel given above, allow us to obtain the estimator of the system's reliability at time k given by

$$\begin{aligned} \widehat{R}(k, M) \quad &:= \quad \alpha_1 \cdot \widehat{\mathbf{P}}_{11}(k, M) \cdot \mathbf{1}_{s_1} \\ &= \quad \alpha_1 \Big[\Big(\delta I - \widehat{\mathbf{q}}_{11}\Big)(\cdot, M) * \Big(I - diag(\widehat{\mathbf{Q}}(\cdot, M) \cdot \mathbf{1})_{11}\Big)\Big](k)\mathbf{1}_{s_1}. \end{aligned} \tag{22}$$

Let us give now the result concerning the consistency and the asymptotic normality of the reliability estimator.

Theorem 2 *For any fixed arbitrary positive integer $k \in \mathbb{N}$, the estimator of the reliability of a discrete-time semi-Markov system at instant k is strongly consistent, i.e.,*

$$\left|\widehat{R}(k, M) - R(k)\right| \xrightarrow[M\to\infty]{a.s.} 0$$

and asymptotically normal, i.e., we have

$$\sqrt{M}[\widehat{R}(k, M) - R(k)] \xrightarrow[M\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma_R^2(k)),$$

with the asymptotic variance

$$\begin{aligned}\sigma_R^2(k) &= \sum_{i=1}^{s} \mu_{ii} \Big\{ \sum_{j=1}^{s} \Big[D_{ij}^U - \mathbf{1}_{\{i\in U\}} \sum_{t\in U} \alpha(t) \Psi_{ti} \Big]^2 * q_{ij}(k) \\ &\quad - \Big[\sum_{j=1}^{s} \Big(D_{ij}^U * q_{ij} - \mathbf{1}_{\{i\in U\}} \sum_{t\in U} \alpha(t) \psi_{ti} * Q_{ij} \Big) \Big]^2 (k) \Big\}, \qquad (23)\end{aligned}$$

where

$$D_{ij}^U := \sum_{n\in U} \sum_{r\in U} \alpha(n) \psi_{ni} * \psi_{jr} * \Big(I - diag(\mathbf{Q} \cdot \mathbf{1}) \Big)_{rr}, \qquad (24)$$

$$\psi(k) := \sum_{n=0}^{k} \mathbf{q}^{(n)}(k), \quad \Psi_{ij}(k) := \sum_{n=0}^{k} Q_{ij}^{(n)}(k), \qquad (25)$$

μ_{ii} – *the mean recurrence time of the state i for the chain Z.* (26)

4.3 Availability Estimation

Taking into account the expression of the availability presented in (11), we propose the following estimator for the availability of a discrete-time semi-Markov system:

$$\begin{aligned}\widehat{A}(k, M) &:= \alpha\, \widehat{\mathbf{P}}(k, M)\, \mathbf{1}_{s,s_1} \\ &= \alpha \Big[\Big(\delta I - \widehat{\mathbf{q}}_{11} \Big) * \Big(I - diag(\widehat{Q}(\cdot, M) \cdot \mathbf{1}) \Big) \Big](k)\, \mathbf{1}_{s,s_1}, \qquad (27)\end{aligned}$$

The following result concerns the consistency and the asymptotic normality of the reliability estimator. A proof of it can be found in [6].

Theorem 3 *For any fixed arbitrary positive integer $k \in \mathbb{N}$, the estimator of the availability of a discrete-time semi-Markov system at instant k is strongly consistent and asymptotically normal, in the sense that*

$$|\, \widehat{A}(k, M) - A(k)\, | \xrightarrow[M\to\infty]{a.s.} 0$$

and

$$\sqrt{M}[\widehat{A}(k, M) - A(k)] \xrightarrow[M\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma_A^2(k)),$$

with the asymptotic variance

$$\sigma_A^2(k) = \sum_{i=1}^{s} \mu_{ii} \Big\{ \sum_{j=1}^{s} \Big[D_{ij} - \mathbf{1}_{\{i\in U\}} \sum_{t=1}^{s} \alpha(t)\Psi_{ti} \Big]^2 * q_{ij}(k) - \Big[\sum_{j=1}^{s} \Big(D_{ij} * q_{ij} - \mathbf{1}_{\{i\in U\}} \sum_{t=1}^{s} \alpha(t)\psi_{ti} * Q_{ij} \Big) \Big]^2 (k) \Big\}, \quad (28)$$

where (29)

$$D_{ij} := \sum_{n=1}^{s} \sum_{r\in U} \alpha(n)\psi_{ni} * \psi_{jr} * \Big(I - diag(Q\cdot \mathbf{1}) \Big)_{rr}.$$

4.4 Failure Rate Estimation

Let us introduce the following notation.

Notation. For a matrix function $A \in \mathcal{M}_E(\mathbb{N})$, we denote by $A^+ \in \mathcal{M}_E(\mathbb{N})$ the matrix function defined by $A^+(k) := A(k+1),\ k \in \mathbb{N}$.

Using the expression of the failure rate obtained in (12), we obtain the following estimator:

$$\widehat{\lambda}(k,M) := \begin{cases} 1 - \frac{\widehat{R}(k,M)}{\widehat{R}(k-1,M)}, & \widehat{R}(k-1,M) \neq 0, \\ 0, & otherwise, \end{cases}$$

$$\widehat{\lambda}(0,M) := 1 - \widehat{R}(0,M),$$

For the failure rate estimator we have similar results as for reliability and availability estimators. A proof of it can be found in [3] and [6].

Theorem 4 *For any fixed arbitrary positive integer $k \in \mathbb{N}$, the estimator of the failure rate of a discrete-time semi-Markov system at instant k is strongly consistent and asymptotically normal, i.e.,*

$$\mid \widehat{\lambda}(k,M) - \lambda(k) \mid \xrightarrow[M\to\infty]{a.s.} 0$$

and

$$\sqrt{M}[\widehat{\lambda}(k,M) - \lambda(k)] \xrightarrow[M\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma_\lambda^2(k)),$$

with the asymptotic variance

$$\sigma_\lambda^2(k) = \frac{1}{R^4(k-1)} \sigma_1^2(k),$$

$$\begin{aligned} \sigma_1^2(k) = \sum_{i=1}^{s} \mu_{ii} \Big\{ & R^2(k) \sum_{j=1}^{s} \Big[D_{ij}^U - \mathbf{1}_{\{i\in U\}} \sum_{t\in U} \alpha(t)\Psi_{ti} \Big]^2 * q_{ij}(k-1) \\ & + R^2(k-1) \sum_{j=1}^{s} \Big[D_{ij}^U - \mathbf{1}_{\{i\in U\}} \sum_{t\in U} \alpha(t)\Psi_{ti} \Big]^2 * q_{ij}(k) - T_i^2(k) \\ & + 2R(k-1)R(k) \sum_{j=1}^{s} \Big[\mathbf{1}_{\{i\in U\}} D_{ij}^U \sum_{t\in U} \alpha(t)\Psi_{ti}^+ + \mathbf{1}_{\{i\in U\}} (D_{ij}^U)^+ \sum_{t\in U} \alpha(t)\Psi_{ti} \\ & - (D_{ij}^U)^+ D_{ij}^U - \mathbf{1}_{\{i\in U\}} \Big(\sum_{t\in U} \alpha(t)\Psi_{ti} \Big) \Big(\sum_{t\in U} \alpha(t)\Psi_{ti}^+ \Big) \Big] * q_{ij}(k-1) \Big\}, \end{aligned} \quad (30)$$

where

$$
\begin{aligned}
T_i(k) \quad &:= \quad \sum_{j=1}^{s} \Big[R(k) D_{ij}^{U} * q_{ij}(k-1) - R(k-1) D_{ij}^{U} * q_{ij}(k) \\
&- \quad R(k) \mathbf{1}_{\{i \in U\}} \sum_{t \in U} \alpha(t) \psi_{ti} * Q_{ij}(k-1) + R(k-1) \mathbf{1}_{\{i \in U\}} \sum_{t \in U} \alpha(t) \psi_{ti} * Q_{ij}(k) \Big], \\
D_{ij}^{U} \quad &- \quad \textit{is given in Equation (24).}
\end{aligned}
$$

4.5 Asymptotic Confidence Intervals

The previously obtained asymptotic results allow one to construct the asymptotic confidence intervals for reliability, availability and failure rate. For this purpose, we need to construct a consistent estimator of the asymptotic variances.

Firstly, using relation (25), we can construct estimators of $\psi(k)$ and of $\Psi(k)$. One can check that these estimators are strongly consistent. Secondly, for $k \leq M$, replacing $q(k), Q(k), \psi(k), \Psi(k)$ respectively by $\widehat{q}(k, M)$, $\widehat{Q}(k, M)$, $\widehat{\psi}(k, M)$, $\widehat{\Psi}(k, M)$ in Equation (23), we obtain an estimator $\widehat{\sigma}_R^2(k)$ of the variance $\sigma_R^2(k)$. From the strong consistency of the estimators $\widehat{q}(k, M), \widehat{Q}(k, M)$, $\widehat{\psi}(k, M)$ and $\widehat{\Psi}(k, M)$ (see [2, 6]), we obtain that $\widehat{\sigma}_R^2(k)$ converges almost surely to $\sigma_R^2(k)$, as M tends to infinity. Finally, the asymptotic confidence interval of $R(k)$ at level $100(1-\gamma)\%$, $\gamma \in (0, 1)$, is:

$$
\widehat{R}(k, M) - u_{1-\gamma/2} \frac{\widehat{\sigma}_R(k)}{\sqrt{M}} \leq R(k) \leq \widehat{R}(k, M) + u_{1-\gamma/2} \frac{\widehat{\sigma}_R(k)}{\sqrt{M}}, \tag{31}
$$

where u_γ is the $\gamma-$ fractile of an $N(0, 1)$- distributed variable. In the same way, we obtain the other confidence intervals.

5 Numerical Example

In this section we apply the previous results to a three-state discrete-time semi-Markov process described in Figure 2. Note that we study here a strictly semi-Markov system, which cannot be reduced to a Markov one.

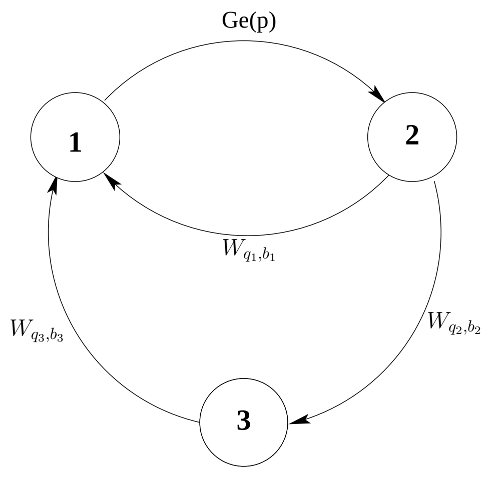

Figure 2: A three-state discrete-time semi-Markov system

Let us consider that the state space $E = \{1, 2, 3\}$ is partitioned into the up-state set $U = \{1, 2\}$ and the down-state set $D = \{3\}$. The system is defined by the initial distribution $\alpha := (1\ 0\ 0)$, by the transition probability matrix $\mathbf{p}$ of the embedded Markov chain $(J_n)_{n\in\mathbb{N}}$ and by the conditional distributions of the sojourn time:

$$\mathbf{p} := \begin{pmatrix} 0 & 1 & 0 \\ 0.95 & 0 & 0.05 \\ 1 & 0 & 0 \end{pmatrix} \quad \mathbf{f}(k) := \begin{pmatrix} 0 & f_{12}(k) & 0 \\ f_{21}(k) & 0 & f_{23}(k) \\ f_{31}(k) & 0 & 0 \end{pmatrix},\ k \in \mathbb{N}.$$

We consider the following distributions for the conditional sojourn time:

- f_{12} is a geometric distribution defined by

  $$f_{12}(0) := 0,\ f_{12}(k) := p(1-p)^{k-1}, k \geq 1,$$

  where we take $p = 0.8$.

- $f_{21} := W_{q_1,b_1}, f_{23} := W_{q_2,b_2}$ and $f_{31} := W_{q_3,b_3}$ are discrete-time, first type Weibull distributions (see [9]), defined by

  $$W_{q,b}(0) := 0,\ W_{q,b}(k) := q^{(k-1)^b} - q^{k^b}, k \geq 1,$$

  where we take $q_1 = 0.3, b_1 = 0.5, q_2 = 0.5, b_2 = 0.7, q_3 = 0.6, b_3 = 0.9$.

Using the transition probability matrix and the sojourn time distributions given above, we have simulated a sample path of the three state semi-Markov chain, of length M. This sample path allows us to compute $N_i(M), N_{ij}(M)$ and $N_{ij}(k, M)$, using Definition 10, and to obtain the empirical estimators $\widehat{p}_{ij}(M)$, $\widehat{f}_{ij}(k, M)$ and $\widehat{q}_{ij}(k, M)$, from relations $(15-17)$. Consequently, we can obtain the estimators $\widehat{Q}(k, M)$, $\widehat{\psi}(k, M)$ and $\widehat{\Psi}(k, M)$. Thus, from Equation (22), we obtain the estimator of the reliability. In Theorem 2 we have obtained the expression of the asymptotic variance of reliability. Replacing $q(k), Q(k), \psi(k), \Psi(k)$ by their estimators in Equation (23), we have the estimator $\widehat{\sigma}_R^2(k, M)$ of the asymptotic variance $\sigma_R^2(k)$. This estimator will allow us to have the asymptotic confidence interval for reliability given in Equation (31).

The consistency of the reliability estimator is illustrated in Figure 3, where the reliability estimators obtained for several values of the sample size M are drawn. In Figure 4 we present the confidence interval of the reliability. Note that the confidence interval covers the true value of the reliability. In Figure 5 we present the estimators of the asymptotic variance of the reliability $\sigma_R^2(k)$, obtained for different sample sizes. We can note that the estimator approaches the true value, as the sample size M increases.

The same type of figures are drawn for the availability and BMP-failure rate. So, in Figures 6, 7 and 8 we have illustrated the consistency of the availability estimator, its asymptotic normality and the consistency of the estimator of the asymptotic variance $\sigma_A^2(k)$. Figures 9, 10 and 11 present the same graphics for the failure rate estimator.

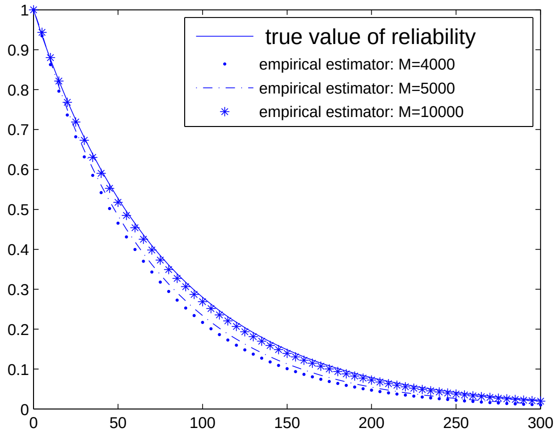

Figure 3: Consistency of reliability estimator

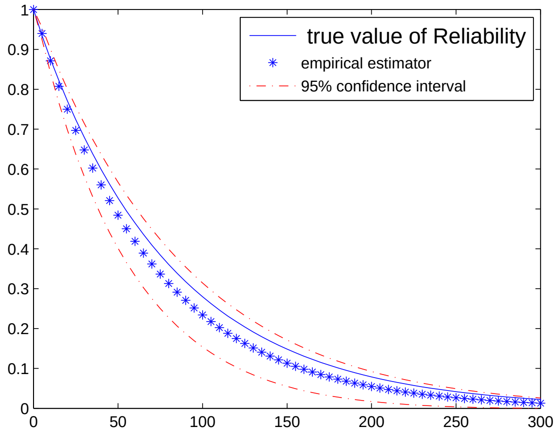

Figure 4: Confidence interval of reliability

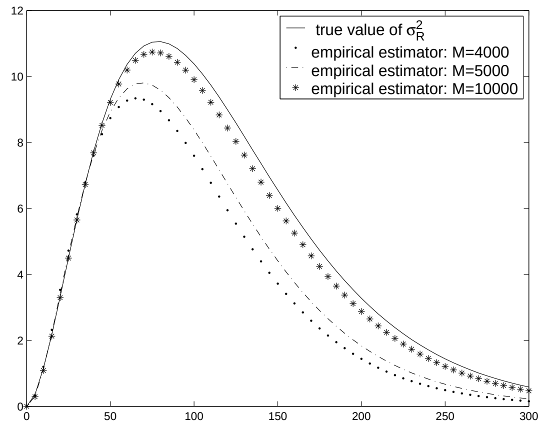

Figure 5: Consistency of $\sigma_R^2(k)$ estimator

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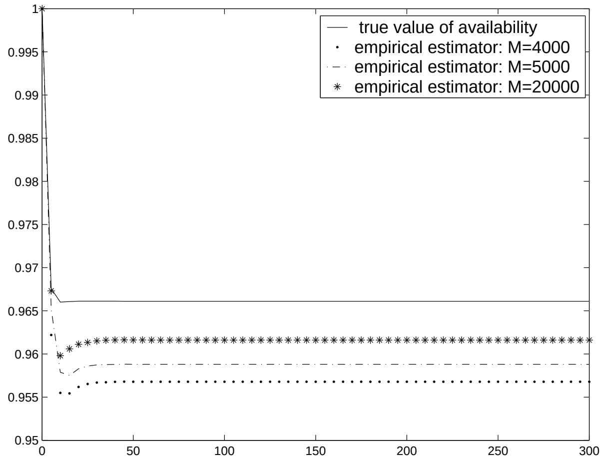

Figure 6: Consistency of availability estimator

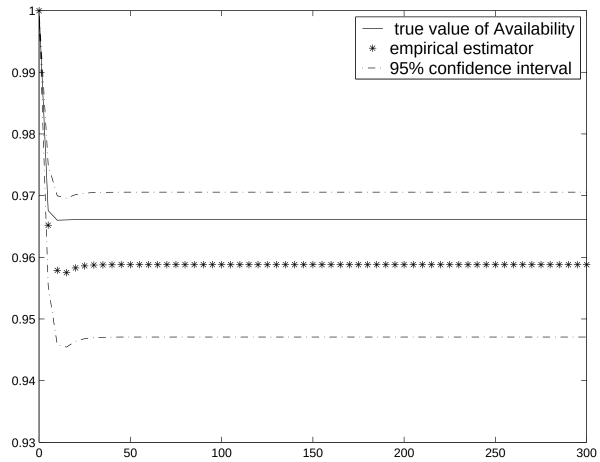

Figure 7: Confidence interval of availability

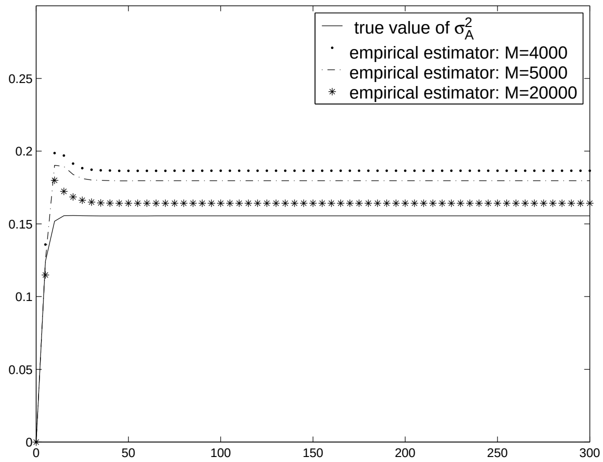

Figure 8: Consistency of $\sigma_A^2(k)$ estimator

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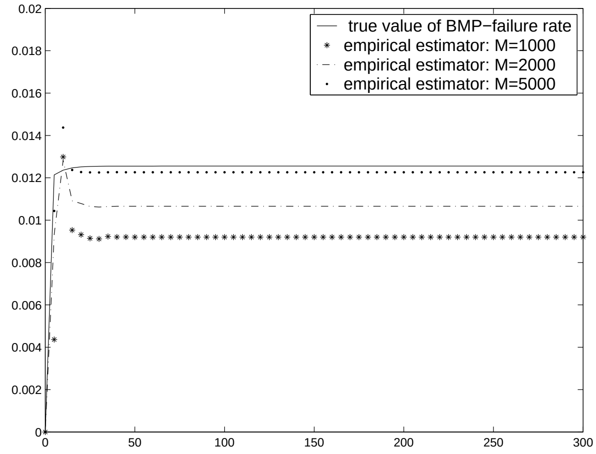

Figure 9: Consistency of failure rate estimator

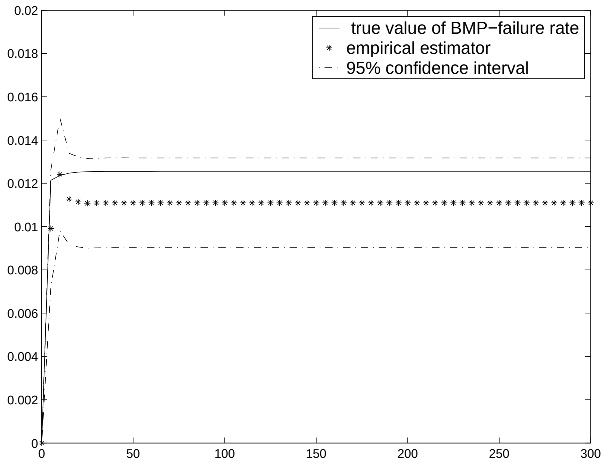

Figure 10: Confidence interval of failure rate

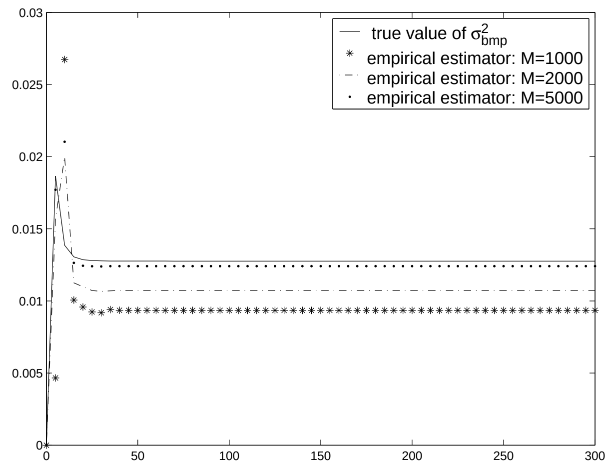

Figure 11: Consistency of $\sigma^2_\lambda(k)$ estimator

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