RANDOM EVOLUTIONS[1]

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Abstract. This article gives a short presentation of random evolutions. At first, the following two examples are presented: dynamical stochastic systems and increment processes both in Markov media. After, an introduction to semi-Markov Random evolution in a Banach space is given, where the previous evolutionary systems are obtained as particular cases. Finally, two abstract limit theorems of average and diffusion approximation for continuous and jump semi-Markov random evolutions are presented. Remarks on bibliography further topics related to random evolutions are given.

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1. INTRODUCTION

Random evolution or stochastic evolution were introduced by R. Hersh and R.J. Griego in 1969. They observed that certain abstract Cauchy problems are related to generators of Markov processes. The name of *Random evolution* is due to P. Lax. The same kind of processes was essentially introduced by R.Z. Khasminskii (1966).

[1]This article will be published in the **Encyclopedia of Statistics**, Wiley.

An evolutionary system that change its mode of evolution following a stochastic mode is a random evolution. For example, a particle moving on the real line with constant velocity, say V, changes it at a random time, given by the arrival of a Poisson process to $-V$, that is, the velocity of the particle at time t is $(-1)^{N(t)}V$, where $N(t), t \geq 0$, is a Poisson process on $[0, \infty)$.

Random evolutions are models to study dynamical systems in random media (environment). In the above example, the Poisson process represent the random medium.

From a mathematical point of view, random evolutions are operator-valued stochastic processes in a Banach space. They represent in an abstract form stochastic evolutionary systems in random media.

A random evolution is an operator valued stochastic process, $\Phi(s, t)$ say, satisfying the following linear differential equation

$$\frac{d}{dt}\Phi(s,t) = -\mathbf{C}(x(t))\Phi(s,t), \quad \Phi(s,s) = \mathbf{I} \tag{1}$$

where operator $\mathbf{C}(x)$ is switched by the stochastic process $x(t), t \geq 0$.

If $x(t), t \geq 0$, is a Markov process with generator $\mathbf{Q}$, then $u(t,x) := \mathbb{E}[\Phi(0,t) \mid x(0) = x]$, satisfies the following equation

$$\frac{du}{dt} = \mathbf{C}(x)u + \mathbf{Q}u. \tag{2}$$

The random evolution, solution of equation (1), verifies the following equation

$$\Phi(s,t) = \Phi(s,u)\Phi(u,t), \quad 0 \leq s \leq u \leq t.$$

This fact was established by Pinsky [26]. So, Φ is also called a *multiplicative operator functional.*

In random evolution theory one of the most important facts is to get limit theorems, where two kind of theorems are obtained in general: a first order and a second order asymptotic corresponding to law of large numbers and to central limit theorems in probability theory. For example, the first order asymptotic scheme for equation (2) is

$$\frac{du_\varepsilon}{dt} = \mathbf{C}(x)u_\varepsilon + \varepsilon^{-1}\mathbf{Q}u_\varepsilon,$$

and the second order asymptotic is

$$\frac{du_\varepsilon}{dt} = \varepsilon^{-1}\mathbf{C}(x)u_\varepsilon + \varepsilon^{-2}\mathbf{Q}u_\varepsilon,$$

where the corresponding limits are obtained as $\varepsilon \to 0$. These are abstract form limit theorems as presented in Section 4.

Applications of random evolutions include: population dynamics in random environment (Cohen (1984)), (Ethier and Kurtz (1987); in insurance (Swishchuk(1999)); reliability analysis (Koroliuk and Limnios (2005)); mathematical physics (Jefferies (1996)); etc.

Discrete-time random evolutions were studied (Keepler (1998)) in connexion with Markov chains, and as embedded random evolution in continuous-time semi-Markov process (Koroliuk and Swishchuk (1995), Koroliuk and Limnios (2005)).

In the next section particular evolutionary systems in Markov media - dynamical and increment processes- and their abstract models in terms of random evolutions are presented. In Section 3, a more general semi-Markov evolution together with the operator Markov renewal equation is presented. In Section 4, two kind of abstract limit theorems for semi-Markov random evolution are presented. Finally, in Section 5, some remarks on the literature and farther topics connected to random evolution are presented.

2. EVOLUTIONARY SYSTEMS IN MARKOV RANDOM MEDIUM

As an introduction to random evolution, two particular stochastic systems switched by Markov processes, and where both are useful in reliability modeling, are given here.

The first one concerns dynamical systems used as a basic model in dynamic reliability (see, e.g., Devooght(1997)); and the second one concerns increment processes used also in reliability of systems suffered shocks and as part in risk process.

Let $(E, \mathcal{E})$ be a standard phase space, that is, E a Polish space, and $\mathcal{E}$ its Borel σ-algebra, and $\mathbf{B}$ a separable Banach space with supremum norm and $\mathcal{B}$ its Borel σ-algebra. Let us consider a jump Markov process $x(t), t \geq 0$, on $(E, \mathcal{E})$, with generator $\mathbf{Q}$, that is,

$$\mathbf{Q}\varphi(x) = \int_E Q(x, dy)[\varphi(y) - \varphi(x)],$$

where $Q(x, dy)$ is the Markov kernel, and a family of generators $\mathbf{\Gamma}(x), x \in E$, that generate a family of strongly continuous contraction semigroups $\mathbb{\Gamma}_x(t), t \geq 0, x \in E$, and the family of bounded linear operators $\mathbf{D}(x, y), x, y \in E$, all acting on the same Banach space $\mathbf{B}$.

Let us consider now the following dynamical system $U(t), t \geq 0$, in the Markov random media $x(t), t \geq 0$, with initial value u,

$$\left| \begin{array}{l} \frac{d}{dt}U(t) = C(U(t); x(t)), \\ \\ U(0) = u. \end{array} \right. \tag{3}$$

The system $U(t)$ takes values into the Euclidean space $\mathbb{R}^d$ and the function $C(u; x)$ defined on $\mathbb{R}^d \times E$ with values into $\mathbb{R}^d$ satisfies the conditions for global solution $U_x(t)$ on $\{x(t) = x\}$ for any $x \in E$.

For any fixed $x \in E$, let us consider the semigroup of operators $\mathbb{C}_t(x), x \in E$, defined by

$$\mathbb{C}_t(x)\varphi(u) = \varphi(U_x(t)), \quad U_x(0) = u. \tag{4}$$

The semigroup property is

$$\mathbb{C}_{t+s}(x) = \mathbb{C}_t(x)\mathbb{C}_s(x), \quad t, s \geq 0, x \in E. \tag{5}$$

The generators $\mathbf{C}(x), x \in E$, of the semigroups $\mathbb{C}_t(x), t \geq 0, x \in E$, are defined by

$$\mathbf{C}(x)\varphi(u) = C(u; x)\varphi'(u), \tag{6}$$

where $\varphi'(u)$ is the derivative of $\varphi(u)$. Of course, product of vectors here means the usual vector scalar product.

The domain of definition of generator $\mathbf{C}(x)$ is the set $C^1(E)$ of continuously differentiable functions defined on E. The generator of the coupled Markov process $U(t), x(t), t \geq 0$, is

$$\mathbf{L} = \mathbf{Q} + \mathbf{C}(x).$$

Let us now consider another evolutionary system, the so-called *increment process* defined as follows

$$\alpha(t) = \sum_{k=1}^{\nu(t)} a(x_k), \tag{7}$$

where a is a measurable real-valued function defined on E, $x_n, n \geq 0$, is the embedded Markov chain of the Markov process $x(t), t \geq 0$ and $\nu(t), t \geq 0$, is the number of jumps of $x(t)$ in the time interval $(0, t]$. In case where the switching process $x(t)$ is Markov, the generator of the coupled process $\alpha(t), x(t), t \geq 0$, is given by

$$\mathbf{L} = \mathbf{Q} + \mathbf{Q}_0[\mathbf{A}(x) - \mathbf{I}], \tag{8}$$

where the operators $\mathbf{Q}_0$ and $\mathbf{A}(x)$, on $\mathbf{B}$, are defined by

$$\mathbf{Q}_0\varphi(x) = \int_E Q(x, dy)\varphi(y), \tag{9}$$

and

$$\mathbf{A}(x)\varphi(u, x) = \varphi(u + a(x), x). \tag{10}$$

The operators $\mathbf{Q}$ and $\mathbf{I}$ are respectively the generator of the Markov process $x(t)$ and the identity operator on $\mathbf{B}$.

Now the random evolutions corresponding to the above two stochastic systems are as follows.

The random evolution $\Phi(t), t \geq 0$, with values in $\mathbf{B}$, corresponding to the dynamical system (3) is the following one

$$\Phi(t)\varphi(u) = \mathbb{C}_{t-\tau_{\nu(t)}}(x(t)) \prod_{k=1}^{\nu(t)} \mathbb{C}_{\theta_k}(x_{k-1})\varphi(u) = \varphi(U(t)), \tag{11}$$

where $0 = \tau_0 < \tau_1 < ... < \tau_n < ...$ are the jumps times, and $\nu(t)$ the number of jumps in $(0, t]$ of $x(t), t \geq 0$.

The random evolution (11) satisfies the following evolutionary equation

$$\frac{d}{dt}\Phi(t) = \mathbf{C}(x(t))\Phi(t) \tag{12}$$

which can be considered as a definition of the random evolution.

The random evolution $\Phi(t), t \geq 0$, corresponding to the increment process (7) can be defined also as follows

$$\Phi(t) = \sum_{k=1}^{\nu(t)} [\mathbf{A}(x_k) - \mathbf{I}]\Phi(\tau_k-), \tag{13}$$

or, the embedded jumps discrete random evolution

$$\Phi(\tau_n) = [\mathbf{A}(x_n) - \mathbf{I}]\Phi(\tau_n-), \quad n \geq 0, \quad \Phi(\tau_0) = \Phi(0) = \mathbf{I}. \tag{14}$$

3. SEMI-MARKOV RANDOM EVOLUTION

Let $x(t), t \geq 0$, be an $(E, \mathcal{E})$-valued semi-Markov process with semi-Markov kernel $Q(x, B, t), x \in E, B \in \mathcal{E}, t \geq 0$. Let $x_n, \tau_n, n \geq 0$, be the embedded Markov renewal process of the semi-Markov process $x(t), t \geq 0$, where $0 = \tau_0 \leq \tau_1 \leq ... \leq \tau_n \leq ...$ are the jump times, and $x_n = x(\tau_n), n \geq 0$, is the embedded $(E, \mathcal{E})$-valued Markov chain with transition kernel $P(x,B), x \in E, B \in \mathcal{E}$, and transition operator $\mathbf{P}$. Let $\nu(t) := \sup\{n \geq 0 : \tau_n \leq t\}$ the number of jumps in the time-interval $(0,t]$. We will suppose also that the semi-Markov process is regular, that is, $\mathbb{P}_x(\nu(t) < \infty) = 1$, for any $x \in E$ and $t > 0$ (see, e.g., [23]). Define also the backward recurrence time process $\gamma(t), t \geq 0$, by $\gamma(t) := t - \tau_{\nu(t)}$, $t \geq 0$. To be specific,

$$Q(x,B,t) := P(x,B)F_x(t), \quad x \in E, B \in \mathcal{E}, t \geq 0, \tag{15}$$

where

$$P(x,B) = \mathbb{P}(x_{n+1} \in B \mid x_n = x), \quad F_x(t) = \mathbb{P}(\theta_{n+1} \leq t \mid x_n = x),$$

where $\theta_{n+1} := \tau_{n+1} - \tau_n, n \geq 0$.

Definition 1. A random evolution is defined as an operator-valued stochastic process $\Phi(t), t \geq 0$, on a Banach space $\mathbf{B}$ that has the following representation

$$\Phi(t) = \mathbb{\Gamma}_{x(t)}(\gamma(t)) \prod_{k=1}^{\nu(t)} \mathbf{D}(x_{k-1}, x_k) \mathbb{\Gamma}_{x_{k-1}}(\theta_k). \tag{16}$$

The random evolution can be represented equivalently by the following integral equation

$$\Phi(t) = \mathbf{I} + \int_0^t \mathbf{\Gamma}(x(s))\Phi(s)ds + \sum_{k=1}^{\nu(t)} [\mathbf{D}(x_{k-1}, x_k) - \mathbf{I}]\Phi(\tau_k -). \tag{17}$$

Definition 2. A *continuous random evolution* in the case where $\mathbf{D}(x,y) = \mathbf{I}$, for all $x, y \in E$, is defined by the solution of the integral equation

$$\Phi_C(t) = \mathbf{I} + \int_0^t \mathbf{\Gamma}(x(s))\Phi_C(s)ds. \tag{18}$$

This equation is equivalent to the above Cauchy problem (12) with $\Phi(0) = \mathbf{I}$.

Definition 3. A *jump random evolution* in the case where $\mathbf{\Gamma}(x) = 0$, for all $x \in E$, is defined as a solution of the following equation

$$\Phi_J(t) = \sum_{k=1}^{\nu(t)} [\mathbf{D}(x_{k-1}, x_k) - \mathbf{I}]\Phi_J(\tau_k -). \tag{19}$$

A *discrete random evolution* $\Phi_n, n \geq 0$, is now defined by

$$\Phi_n = \prod_{k=1}^{n} [\mathbf{D}(x_{k-1}, x_k) - \mathbf{I}] \Gamma_{x_{k-1}}(\theta_k), \quad n \geq 0, \quad \Phi_0 = \mathbf{I}. \tag{20}$$

Of course $\Phi_n := \Phi(\tau_n)$, $n \geq 0$.

The mean value of the random evolution $\Phi(t)\varphi$, that is,

$$U(t,x) := \mathbb{E}_x[\Phi(t)\varphi(x(t))], \tag{21}$$

verifies the *operator Markov renewal equation* for random evolution

$$U(t,x) - \int_0^t \int_E Q(x,dy,ds)\mathbf{D}(x,y)\Gamma_x(s)U(t-s,y) = \overline{F}_x(t)\Gamma_x(t)\varphi(x). \tag{22}$$

Let $\xi_n, n \geq 0$, be the embedded stochastic system, for example, $\xi_n = U(\tau_n)$, $n \geq 0$. The extended *compensating operator* of the extended Markov renewal process $\xi_n, x_n, \tau_n, n \geq 0$, is defined by

$$\mathbf{L}\varphi(u,x,t) := q(x)\{\mathbb{E}[\varphi(\xi_n, x_{n+1}, \tau_{n+1}) \mid \xi_n = u, x_n = x, \tau_n = t] - \varphi(u,x,t)\}, \tag{23}$$

where $q(x) := 1/m(x)$ and $m(x) := \int_0^\infty \overline{F}_x(t)dt$, $\overline{F}_x(t) = 1 - F_x(t)$, $x \in E$.

From the above definition, the following analytic form of the compensating operator is easily obtained

$$\mathbf{L}\varphi(u,x,t) = q(x)\Big[\int_0^\infty \int_E Q(x,dy,ds)\mathbf{D}(x,y)\Gamma_x(s)\varphi(u,y,t+s) - \varphi(u,x,t)\Big]. \tag{24}$$

4. ABSTRACT LIMIT THEOREMS

Let $x(t), t \geq 0$, be the semi-Markov process considered in the previous section. We suppose that the associated Markov process $x_0(t), t \geq 0$, defined by the generator

$$\mathbf{Q}\varphi(x) = q(x)\int_E P(x,dy)[\varphi(y) - \varphi(x)],$$

is uniformly ergodic with stationary distribution $\pi(dx)$, where $q(x)$ is the jumps intensity function. Let $\mathbf{\Pi}$ be the projector on the Banach space $\mathbf{B}$, of bounded measurable functions on E, defined by

$$\mathbf{\Pi}\varphi(x) = \int_E \pi(dy)\varphi(y)\mathbf{1}(x),$$

and the potential operator $\mathbf{R}_0$ of the associated ergodic Markov process $x_0(t)$, which is defined by

$$\mathbf{R}_0 := \int_0^\infty (\mathbf{P}_t - \mathbf{\Pi})dt,$$

and properties

$$\mathbf{Q}\mathbf{R}_0 = \mathbf{R}_0\mathbf{Q} = \mathbf{\Pi} - \mathbf{I},$$

where the operators $\mathbf{P}_t, t \geq 0$, are the semigroup of $x_0(t)$.

Definition 4. A *continuous semi-Markov random evolution* in *average* series scheme, with a small parameter $\varepsilon > 0$, is defined by

$$\left| \begin{array}{l} \frac{d}{dt}\Phi_C^\varepsilon(t) = \mathbf{\Gamma}(x(t/\varepsilon))\Phi_C^\varepsilon(t), \quad t \geq 0, \\ \\ \Phi_C^\varepsilon(0) = \mathbf{I}. \end{array} \right. \tag{25}$$

Here $\mathbf{\Gamma}(x), x \in E$, is the family of generators of the semigroup operators $\mathbb{\Gamma}_t(x), t \geq 0, x \in E$, which determines the random evolution in the following form

$$\Phi_C^\varepsilon(t) = \mathbb{\Gamma}_{x(t/\varepsilon)}(\varepsilon\gamma(t/\varepsilon)) \prod_{k=1}^{\nu(t/\varepsilon)} \mathbb{\Gamma}_{x_k}(\varepsilon\theta_k), \quad t > 0, \quad \Phi^\varepsilon(0) = \mathbf{I}. \tag{26}$$

Definition 5. The *coupled random evolution* on the Banach space $C(\mathbb{R}^d \times E)$, of real-valued measurable bounded functions, is defined by

$$\Phi^\varepsilon(t; x^\varepsilon(t/\varepsilon)) = \Phi^\varepsilon(t)\varphi(u, x^\varepsilon(t/\varepsilon)). \tag{27}$$

The above random evolution is characterized by the *compensating operator* of the extended Markov renewal process $\zeta_n, x_n^\varepsilon, n \geq 0$,

$$\mathbf{L}^\varepsilon\varphi(u, x) = \varepsilon^{-1}q(x)\Big[\int_0^\infty \int_E Q(x, dy, ds)\mathbb{\Gamma}_x^\varepsilon(\varepsilon s)\varphi(u, y) - \varphi(u, x)\Big]. \tag{28}$$

The truncated part, up to a negligible term, of the asymptotic form of the above compensating operator is

$$\mathbf{L}_0^\varepsilon = \varepsilon^{-1}\mathbf{Q} + \mathbf{\Gamma}(x)\mathbf{P}. \tag{29}$$

By a solution of a singular perturbation problem (see [19]), we get the following result.

Theorem 1. Under the above assumption, the limit operator, as $\varepsilon \to 0$, is

$$\widehat{\mathbf{\Gamma}} = \mathbf{\Pi}\mathbf{\Gamma}(x)\mathbf{\Pi}, \quad \widehat{\mathbf{\Gamma}} = \int_E \pi(dx)\mathbf{\Gamma}(x). \tag{30}$$

The *diffusion approximation scheme* of the continuous random evolution is as follows.

$$\left| \begin{array}{l} \frac{d}{dt}\Phi_C^\varepsilon(t) = \varepsilon^{-1}\mathbf{\Gamma}(x(t/\varepsilon^2))\Phi_C^\varepsilon(t), \quad t \geq 0, \\ \\ \Phi_C^\varepsilon(0) = \mathbf{I}. \end{array} \right. \tag{31}$$

The *coupled random evolution* on $C(\mathbb{R}^d \times E)$

$$\Phi^\varepsilon(t; x^\varepsilon(t/\varepsilon^2)) = \Phi^\varepsilon(t)\varphi(u, x(t/\varepsilon^2)), \quad x(0) = x \tag{32}$$

is characterized by the compensating operator of extended Markov renewal process

$$\mathbf{L}^\varepsilon\varphi(u,x) = \varepsilon^{-2}q(x)\Big[\int_0^\infty \int_E Q(x,dy,ds)\mathbb{\Gamma}_x^\varepsilon(\varepsilon^2 s)\varphi(u,y) - \varphi(u,x)\Big] \tag{33}$$

of which the asymptotic truncated, up to a negligible term, representation is

$$\mathbf{L}_0^\varepsilon = \varepsilon^{-2}\mathbf{Q} + \varepsilon^{-1}\mathbf{\Gamma}(x)\mathbf{P} + \mathbf{Q}_1(x)\mathbf{P}, \tag{34}$$

where $\mathbf{Q}_1(x) = m_2(x)\mathbf{\Gamma}^2(x)/2m(x)$, and $m_2(x) := \int_0^\infty t^2 dF_x(t)$.

Theorem 2. Let us suppose that the square sojourn times $\theta_x^2, x \in E$, are uniformly integrable, and that the balance condition $\mathbf{\Pi}\mathbf{\Gamma}(x)\mathbf{\Pi} = 0$ holds, then the limit operator, as $\varepsilon \to 0$, is

$$\widehat{\mathbf{L}} = \int_E \pi(dx)[\mathbf{\Gamma}(x)\mathbf{R}_0\mathbf{\Gamma}(x) + \mu(x)\mathbf{\Gamma}^2(x)], \tag{35}$$

where $\mu(x) := [m_2(x) - 2m^2(x)]/m(x)$.

The *jump semi-Markov random evolution* in the *average* series scheme is as follows

$$\Phi_J^\varepsilon(t) = \prod_{k=1}^{\nu(t/\varepsilon)} \mathbf{D}^\varepsilon(x_k^\varepsilon), \quad \Phi_J^\varepsilon(0) = \mathbf{I}, \tag{36}$$

where the bounded operators $\mathbf{D}^\varepsilon(x), x \in E$, have the following asymptotic representation

$$\mathbf{D}^\varepsilon(x) - \mathbf{I} = \varepsilon\mathbf{D}(x) + \varepsilon\mathbf{D}_1^\varepsilon(x), \tag{37}$$

on the space $\mathbf{B}_0$, dense in $C_0^2(\mathbb{R}^d \times E)$, with the negligible term

$$\|\mathbf{D}_1^\varepsilon(x)\varphi\| \to 0, \quad \varepsilon \to 0, \quad \varphi \in \mathbf{B}_0.$$

The compensating operator of this jump semi-Markov random evolution is given by

$$\mathbf{L}^\varepsilon\varphi(u,x) = \varepsilon^{-1}q(x)\Big[\int_E P(x,dy)\mathbf{D}^\varepsilon(y)\varphi(u,y) - \varphi(u,x)\Big], \tag{38}$$

where the asymptotic form is

$$\mathbf{L}^\varepsilon = \varepsilon^{-1}\mathbf{Q} + \mathbf{Q}_0\mathbf{D}(x) + \mathbf{Q}_0\mathbf{D}_1^\varepsilon(x). \tag{39}$$

Theorem 3. The limit operator obtained as a solution of a singular perturbation problem on the asymptotic form of the above compensating operator is given by

$$\widehat{\mathbf{L}} := \mathbf{\Pi}\mathbf{Q}_0\mathbf{D}(x)\mathbf{\Pi} = \widehat{\mathbf{D}}\mathbf{\Pi}, \quad \widehat{\mathbf{D}} := q\int_E \rho(dx)\mathbf{D}(x), \tag{40}$$

where $q := 1/m$, $m := \int_E \rho(dx)m(x)$.

In the *diffusion approximation* scheme, the jump semi-Markov random evolution is

$$\Phi^\varepsilon(t) = \prod_{k=1}^{\nu(t/\varepsilon^2)} \mathbf{D}^\varepsilon(x_k^\varepsilon), \quad \Phi^\varepsilon(0) = \mathbf{I}, \tag{41}$$

where the bounded operators $\mathbf{D}^\varepsilon(x), x \in E$, have the following asymptotic representation

$$\mathbf{D}^\varepsilon(x) - \mathbf{I} = \varepsilon\mathbf{D}(x) + \varepsilon^2\mathbf{D}_1(x) + \varepsilon^2\mathbf{D}_2^\varepsilon(x), \tag{42}$$

on the space $\mathbf{B}_0$, dense on the space $C_0^2(\mathbb{R}^d \times E)$, with the negligible term

$$\|\mathbf{D}_2^\varepsilon(x)\varphi\| \to 0, \quad \varepsilon \to 0, \quad \varphi \in \mathbf{B}_0.$$

The compensating operator of this semi-Markov random evolution is

$$\mathbf{L}^\varepsilon\varphi(u,x) = \varepsilon^{-2}q(x)\Big[\int_E P(x,dy)\mathbf{D}^\varepsilon(y)\varphi(u,y) - \varphi(u,x)\Big]. \tag{43}$$

Theorem 4. Under the above assumptions, and the additional balance condition $\widehat{\mathbf{D}} = 0$, the limit operator, as $\varepsilon \to 0$, is

$$\widehat{\mathbf{L}} := \mathbf{\Pi}\mathbf{Q}_0\mathbf{D}_1(x)\mathbf{\Pi} + \mathbf{\Pi}\mathbf{Q}_0\mathbf{D}(x)\mathbf{R}_0\mathbf{Q}_0\mathbf{D}(x)\mathbf{\Pi}. \tag{44}$$

Remark 1. It is worth noticing that when we consider the semi-Markov random evolution with continuous and discrete parts, the limit operator of the whole compensating operator is the sum of the above two limit operators. For example, in the case of average series scheme the limit operator is $\widehat{\mathbf{L}}_0 := \widehat{\mathbf{\Gamma}} + \widehat{\mathbf{D}}$, where $\widehat{\mathbf{\Gamma}}$ and $\widehat{\mathbf{D}}$ are given by relations (30) and (40) respectively.

Remark 2. Relations (35) and (44) are what we need in order to get average and diffusion approximation results respectively for particular systems.

Example. The dynamical system (3) is considered here in the following time-scaling scheme

$$\left| \begin{array}{l} \frac{d}{dt}U^\varepsilon(t) = C(U^\varepsilon(t), x(t/\varepsilon^2)), \quad t \geq 0, \\ \\ U^\varepsilon(0) = u, \end{array} \right. \tag{45}$$

where the semi-Markov family processes $x(t/\varepsilon^2), t \geq 0, \varepsilon > 0$, is as described above, and $U^\varepsilon(t) \in \mathbb{R}^d$. This system usually describes dynamic reliability where the semi-Markov process describes the structure of the system and the process $U^\varepsilon(t)$ the physical throughput in the system, for example, temperature, pressure, velocity, etc. (see, e.g., [5]).

This system can be represented by the following continuous semi-Markov random evolution

$$\Phi^\varepsilon(t)\varphi(u, x(t/\varepsilon^2)) := \varphi(U^\varepsilon(t), x(t/\varepsilon^2)), \tag{46}$$

with the family of semigroups

$$\mathbb{\Gamma}_x(t)\varphi(u) := \varphi(U(t; u, x)), \quad x \in E, t \geq 0, \tag{47}$$

and generators

$$\mathbf{\Gamma}(x)\varphi(u) := C(u, x)\varphi'(u), \quad x \in E. \tag{48}$$

Then under the balance condition

$$\int_E \pi(dx)C(u, x) \equiv 0, \tag{49}$$

the limit generator (35) gives

$$\mathbf{L}\varphi(u) = a(u)\varphi'(u) + \frac{1}{2}B(u)\varphi''(u), \tag{50}$$

for $\varphi \in C^2(\mathbb{R}^d)$, and where the drift vector function is

$$a(u) := \int_E \pi(dx)[C(u,x)\mathbf{R}_0 C'_u(u,x) + \frac{1}{2}\mu(x)C(u,x)C'_u(u,x)],$$

and the diffusion coefficient matrix is

$$B(u) := 2\int_E \pi(dx)[C(u,x)\mathbf{R}_0 C(u,x) + \mu(x)C(u,x)C^*(u,x)],$$

be assumed positive defined. In the Markov case, we have $\mu(x) \equiv 0$.

So, the limit process is a diffusion process with drift $a(u)$ and diffusion coefficient $\sigma(u)$ defined by $B(u) = \sigma(u)\sigma^*(u)$.

5. CONCLUDING REMARKS

Detailed definitions of coupled semi-Markov and Markov random evolutions, as well as of compensating operator and its properties can be found in [19]. Average and diffusion approximation scheme of semi-Markov random evolutions with split and merging of the phase space as well as non ergodic switching processes with applications in reliability can be found in [19]. For more detailed results on random evolutions see books [4, 14, 18, 19, 20, 21, 27], and review papers [9, 10] and references thereby. Detailed proofs of theorems given is this article can be found in [19]. Theorem 1–4 presented here are taken from [19].

Apart average and diffusion approximation considered here, random evolutions can be used also in Poisson and Lévy approximation schemes, where they naturally associated to the predictable characteristics of additive semimartingale approach, see, e.g., [19, ch. 7].

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Further Reading

Related Entries: stochastic process, semigroup, Markov chain, Markov process, semi-Markov process, Markov renewal process, differential equation.