RANDOM EVOLUTIONS TOWARD APPLICATIONS[1]

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Abstract. This article gives a short and elementary presentation of random evolutions toward applications in reliability and quality engineering. At first, the following two examples are presented: dynamical stochastic systems and increment processes both in Markov media. A dynamical system in continuous time is presented since nowadays they are widely used in dynamic reliability modeling. Limit theorems in averaging and diffusion approximation scheme for ergodic and non ergodic media are also presented. Remarks on bibliography on further topics related to random evolutions are given.

Keywords and Phrases. Random evolution, stochastic evolution, dynamical system, Markov process, stochastic process, Markov chain, Markov renewal process, reliability, Piecewise Deterministic Markov process, differential equation.

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1. INTRODUCTION

Random evolution or stochastic evolution were introduced by R. Hersh and R.J. Griego in 1969. They observed that certain abstract Cauchy problems are related to

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generators of Markov processes. The name of *Random evolution* is due to P. Lax.

An evolutionary system that change its mode of evolution following a stochastic mode is a random evolution. For example, a particle moving on the real line with constant velocity, say V, changes it at a random time, given by the arrival of a Poisson process to $-V$, that is, the velocity of the particle at time t is $(-1)^{N(t)}V$, where $N(t), t \geq 0$, is a (homogeneous) Poisson process on $[0, \infty)$.

Random evolutions are models to study stochastic systems in random media (environment). In the above example, the Poisson process represent the random medium.

From a mathematical point of view, random evolutions are operator-valued stochastic processes in a Banach space. They represent an abstract form stochastic evolutionary systems in random media. Discrete-time random evolution where studied in connection with Markov chains (see, e.g., Keepler (1998)), and as embedded random evolution in continuous-time semi-Markov process (see, e.g., Koroliuk and Swishchuk (1995), Koroliuk and Limnios (2005)). In Skorokhod et al. (2002) several random evolution problems are studied.

Applications of random evolutions include: population dynamics in random environment (Cohen (1984)); in insurance (Swishchuk(1999)); reliability analysis (Koroliuk and Limnios (2005)); in population dynamics (Ethier and Kurtz (1987), etc.

In this paper we present random evolutions via dynamical systems, the so-called Piecewise Deterministic Markov processes (see Davis (1993)) in ergodic and non ergodic media.

In the next section a particular dynamical system in Markov media and its abstract model in terms of random evolutions is presented. In Section 3, two types of limit theorems for Markov random evolution are presented. Finally, in Section 4, some remarks on the literature and farther topics connected to random evolution are presented.

2. EVOLUTIONARY SYSTEMS IN MARKOV RANDOM MEDIUM

Let us consider the dynamical system

$$\left| \begin{array}{l} \frac{d}{dt}U(t) = C(U(t); x(t)), \quad t \geq 0, \\ \\ U(0) = u. \end{array} \right. \tag{1}$$

where the process $x(t), t \geq 0$, is a Markov process with state space E, and generator $\mathbf{Q}$, and $U(t) \in \mathbb{R}^d$. This system usualy describes dynamic reliability where the Markov process $x(t), t \geq 0$, describes the structure of the system and the process $U(t)$ the physical throughput in the system, for example, temperature, pressure, velocity, etc. (see, e.g., [5]).

The coupled stochastic process $(U(t), x(t)), t \geq 0$, is a Markov process, with state space $\mathbb{R}^d \times E$ and generator $\mathbf{L}$:

$$\mathbf{L}\varphi(u, x) = \mathbf{\Gamma}(x)\varphi(u, \cdot) + \mathbf{Q}\varphi(\cdot, x),$$

where the operator $\mathbf{\Gamma}(x)$ acts on functions $\varphi(u) \in C^1(\mathbb{R}^d)$ as follows

$$\mathbf{\Gamma}(x)\varphi(u) = C(u; x)(\frac{\partial}{\partial u_1}, ..., \frac{\partial}{\partial u_d})\varphi(u_1, ..., u_d)$$

For example, in a two state Markov process, $x(t)$, say $E = \{0, 1\}$ and $d = 1$, the generator $\mathbf{L}$ can be written in the following matrix form

$$\mathbf{L} = \begin{pmatrix} C_0(u)\frac{\partial}{\partial u} & 0 \\ 0 & C_1(u)\frac{\partial}{\partial u} \end{pmatrix} + \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

where $C_x(u) := C(u; x), x = 0, 1$.

As the trajectory of $U(t)$ at time $t + s$, with initial value $U(0) = u$, is an extension of the trajectory at time s of the trajectory with initial value $U(t; u, x)$ it is easy to see that the solution of equation (1) for fixed state $x \in E$, verifies the semigroup property

$$U(t+s;u,x) = U(s;U(t;u,x),x),$$

where $U(t;u,x)$ is the value of $U(t)$ on $\{U(0)=u, x(0)=x\}$.

An abstract formulation of the above property can be represented by the following continuous Markov random evolution

$$\Phi(t)\varphi(u,x(t)) := \varphi(U(t);x(t)), \tag{2}$$

with the family of semigroups

$$\mathbb{\Gamma}_x(t)\varphi(u) := \varphi(U(t;u,x)), \quad x \in E, t \geq 0, \tag{3}$$

and generators

$$\mathbf{\Gamma}(x)\varphi(u) := C(u,x)\frac{d}{du}\varphi(u), \quad x \in E. \tag{4}$$

3. ASYMPTOTIC RESULTS

Let us consider the dynamical system in (1) in two different series schemes. The first one is the ergodic case for the switching process $x(t), t \geq 0$, and in the second one a split with absorption is considered. In both cases, average diffusion approximation limit theorems will be presented.

Let us consider the switching jump Markov processes $x^\varepsilon(t), t \geq 0, \varepsilon > 0$, in series scheme with series parameter $\varepsilon > 0$ ($\varepsilon \to 0$), with state space $(E, \mathcal{E})$, where E is a Polish space and $\mathcal{E}$ its Borel σ-algebra, and generators

$$Q^\varepsilon\varphi(x) = q(x)\int_E P^\varepsilon(x,dy)[\varphi(y)-\varphi(x)], \tag{5}$$

where $P^\varepsilon(x,dy)$ is the transition kernel of the embedded Markov chain $x_n^\varepsilon = x^\varepsilon(\tau_n^\varepsilon)$ and τ_n^ε are the jump times of $x^\varepsilon(t), t \geq 0$. The function $q(x)$ is the intensity of jumps.

3.1. Ergodic case

Let us first present result on averaging. The following assumptions are needed.

A1. The switching Markov process $x(t), t \geq 0$, is uniformly ergodic with stationary probability measure $\pi(B), B \in \mathcal{E}$.

A2. The function $C(u; x)$ is globally Lipschitz continuous on $u \in \mathbb{R}^d$ with common constant L for all $x \in E$. So, the system

$$\frac{d}{dt} U_x(t) = C(U_x(t); x),$$

has a global solution.

Theorem 1. [Average approximation] Under Assumptions A1 and A2, the solution $U^\varepsilon(t)$ of the following system

$$\left| \begin{array}{l} \frac{d}{dt} U^\varepsilon(t) = C(U^\varepsilon(t); x(t/\varepsilon)) \\ U^\varepsilon(0) = u, \end{array} \right. \tag{6}$$

converges weakly to the solution of the average equation

$$\left| \begin{array}{l} \frac{d}{dt} \widehat{U}(t) = \widehat{C}(\widehat{U}(t)) \\ \widehat{U}(0) = u \end{array} \right.$$

where $\widehat{C}(u) := \int_E \pi(dx) C(u; x)$.

It is worth noticing that this is a deterministic system and this averaging scheme is the so-called Bugolyubov principle.

For the *diffusion approximation* we consider the following time-scalling of the switching process.

$$\left| \begin{array}{l} \frac{d}{dt} U^\varepsilon(t) = C^\varepsilon(U^\varepsilon(t); x(t/\varepsilon^2)) \\ U^\varepsilon(0) = u, \end{array} \right. \tag{7}$$

where $C^\varepsilon(u; x) := \varepsilon^{-1} C(u; x) + C_1(u; x)$.

We suppose here that the following balance condition is fulfilled

$$\widehat{C}(u) = 0. \tag{8}$$

Theorem 2. [Diffusion approximation] Under Assumption A1, and the balance condition (8), the solution of the system (7) converges to a diffusion process, provided that the diffusion coefficient B is > 0, the limit diffusion process is defined by the following infinitesimal generator

$$\mathbf{L}\varphi(u) = \widehat{C}_0\varphi'(u) + \frac{1}{2}B(u)\varphi''(u),$$

where the drift coefficient is

$$\widehat{C}_0(u) = \int_E \pi(dx)[C_1(u;x) + C(u;x)R_0C'_u(u;x)],$$

and the diffusion coefficient is

$$B = 2\int_E \pi(dx)C_0(u;x)R_0C(u;x),$$

Operator R_0 is the potential operator of Q, that is

$$QR_0 = R_0Q = \Pi - I,$$

and Π is the projector operator defined by $\Pi(x, dy) = \pi(dy)$, for every $x \in E$.

Notation φ', φ'' stand for the first and second derivatives of function φ, and $C'_u(u;x)$ stads for the partial derivative of $C(u;x)$ with respect to the variable u.

3.2. Non ergodic case

While the ergodic case is more adapted to availability modeling, the non ergodic case, presented here, is more adapted to reliability modeling.

Let us now consider the following split to the state space of the Markov switching processes $x^\varepsilon(t),\ t \geq 0,\ \varepsilon > 0$,

$$E^0 = E\bigcup\{0\}, \quad E = \bigcup_{k=1}^{N} E_k, \quad E_k\bigcap E_{k'} = \emptyset,\ k \neq k', \tag{9}$$

with absorbing state $\{0\}$. An interpretation of this split in reliability problems is that $E_1, ..., E_N$ are subsets of different performance levels and 0 is the failure state of the system.

Let us suppose that the transition kernel P^ε has the following representation

$$P^\varepsilon(x,B) = P(x,B) + \varepsilon P_1(x,B), \tag{10}$$

where $P(x,B)$ is a transition kernel, coordinated with the above split (9) as follows

$$P(x,E_k) = \mathbf{1}_k(x) := \begin{cases} 1, & x \in E_k \\ 0, & x \notin E_k, \end{cases} \tag{11}$$

and $P_1(x,B)$ a perturbing kernel.

The Markov supporting process $x(t),\ t \geq 0$, on the state space $(E,\mathcal{E})$, determined by the generator

$$Q\varphi(x) = q(x)\int_E P(x,dy)[\varphi(y) - \varphi(x)], \tag{12}$$

is supposed to be uniformly ergodic in every class $E_k,\ 1 \leq k \leq N$, with the stationary distribution $\pi_k(dx),\ 1 \leq k \leq N$, satisfying the following relations

$$\pi_k(dx)q(x) = q_k\rho_k(dx), \quad q_k = \int_{E_k}\pi_k(dx)q(x),$$

$$\rho_k(B) = \int_{E_k}\rho_k(dx)P(x,B), \quad \rho_k(E_k) = 1.$$

where ρ_k is the stationary probability of the embedded Markov chain $x_n, n \geq 0$, of the supporting Markov process $x(t), t \geq 0$, on the class E_k, $1 \leq k \leq N$.

Let v be the merging function, defined by

$$v(x) = k \quad \text{if} \quad x \in E_k, \quad 1 \leq k \leq N.$$

Then we have

$$v(x^\varepsilon(t/\varepsilon)) \Longrightarrow \widehat{x}(t), \quad \varepsilon \to 0.$$

The process $\widehat{x}(t), t \geq 0$ is a Markov process on the state space $E^0 = E \cup \{0\}$, with $\widehat{E} = \{1,...,N\}$.

The random variable ζ^ε is the absorpion time or lifetime of failure time for the system, and is defined by

$$\zeta^\varepsilon := \inf\{t \geq 0 : x^\varepsilon(t/\varepsilon) = 0\}.$$

We have $\zeta^\varepsilon \Longrightarrow \widehat{\zeta}$, as $\varepsilon \to 0$, where $\widehat{\zeta}$ is the absorption time of the merged process $\widehat{x}(t)$.

Theorem 3. [Average approximation with split and merging] Under the above assumptions the stochastic system $U^\varepsilon(t),\ t \geq 0$, defined by

$$\left| \begin{array}{l} \frac{d}{dt}U^\varepsilon(t) = C(U^\varepsilon(t); x^\varepsilon(t/\varepsilon)) \\ U^\varepsilon(0) = u, \end{array} \right. \tag{13}$$

converges weakly to the averaged stochastic system $\widehat{U}(t \wedge \widehat{\zeta})$:

$$U^\varepsilon(t \wedge \zeta^\varepsilon) \Rightarrow \widehat{U}(t \wedge \widehat{\zeta}), \quad \text{as } \varepsilon \to 0.$$

The limit process $\widehat{U}(t),\ t \geq 0$, is defined by a solution of the evolutionary equation

$$\frac{d}{dt}\widehat{U}(t) = \widehat{C}(\widehat{U}(t); \widehat{x}(t)), \quad \widehat{U}(0) = 0,$$

on the time interval $0 \leq t \leq \widehat{\zeta}$, ($\widehat{\zeta}$ is the stoppage time of the merged Markov process $\widehat{x}(t),\ t \geq 0$).

The averaged velocity is determined by

$$\widehat{C}(u; k) = \int_{E_k} \pi_k(dx) C(u; x), \quad 1 \leq k \leq N, \quad \widehat{C}(u; 0) = 0.$$

For the diffusion approximation scheme, we suppose that $N = 1$ for simplicity, and that the transition kernel has the following representation

$$P^\varepsilon(x, B) = P(x, B) + \varepsilon^2 P_1(x, B).$$

The random variable $\widehat{\zeta}$ is exponentially distributed with parameter $\widehat{\Lambda} = pq$, where $p := -\int_E \rho(dx) P_1(x, E)$ and $q := \int_E \rho(dx)/q(x)$.

The considered system here is in the following time-scaling scheme.

$$\left| \begin{array}{l} \frac{d}{dt}U^\varepsilon(t) = C^\varepsilon(U^\varepsilon(t); x^\varepsilon(t/\varepsilon^2)) \\ U^\varepsilon(0) = u, \end{array} \right. \tag{14}$$

where $C^\varepsilon(u; x) := \varepsilon^{-1}C(u; x) + C_1(u; x)$.

Theorem 4. [Diffusion approximation with split and merging] Under the balance condition (8), the stochastic system (14) converges weakly, as $\varepsilon \to 0$,

$$U^\varepsilon(t \wedge \zeta^\varepsilon) \Rightarrow \xi(t \wedge \widehat{\zeta}) \ \text{ as } \varepsilon \to 0.$$

The limit diffusion process $\xi(t),\ t \geq 0$, is defined by the generator

$$\mathbf{L}\varphi(u) = b(u)\varphi'(u) + \frac{1}{2}B(u)\varphi''(u) - \widehat{\Lambda}\varphi(u). \tag{15}$$

The drift coefficient is defined by

$$b(u) = \int_E \pi(dx)[C_1(u; x) + C(u; x)R_0C'_u(u; x)].$$

The covariance function is defined by

$$B(u) = 2\int_E \pi(dx)C(u; x)R_0C(u; x).$$

The parameter $\widehat{\Lambda}$ in (15) is the parameter of the exponential distribution of the lifetime of the merged Markov process and of the limit diffusion process too.

4. CONCLUDING REMARKS

Detailed definitions of random evolutions, in semi-Markov and Markov media, can be found in [20]. Average and diffusion approximation scheme of semi-Markov random evolutions with split and merging of the phase space as well as non ergodic switching processes with applications in reliability can be found in [20]. For more references on random evolutions see books [27, 22, 21, 19, 20], and review paper [10] and references thereby.

A part average and diffusion approximation considered here, random evolutions can be used also in Poisson and Lévy approximation schemes, where they are naturally associated to the predictable characteristics in additive semimartingale approach, see, e.g., [20, ch. 7].

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Further Reading

Related Entries: stochastic process, semigroup, Markov chain, Markov process, semi-Markov process, Markov renewal process, differential equation.