# Introduction to belief functions

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# Contents of this lecture

1. Historical perspective, motivations
2. Fundamental concepts: belief, plausibility, commonality, conditioning, basic combination rules
3. Some more advanced concepts: cautious rule, multidimensional belief functions, belief functions in infinite spaces

# Uncertain reasoning

- In science and engineering we always need to reason with partial knowledge and uncertain information (from sensors, experts, models, etc.)
- Different sources of uncertainty
  - Variability of entities in populations and outcomes of random (repeatable) experiments $\rightarrow$ Aleatory uncertainty. Example: drawing a ball from an urn. Cannot be reduced
  - Lack of knowledge $\rightarrow$ Epistemic uncertainty. Example: inability to distinguish the color of a ball because of color blindness. Can be reduced
- Classical ways of representing uncertainty
  1. Using probabilities
  2. Using set (e.g., interval analysis), or propositional logic

# Probability theory

- Probability theory can be used to represent
  - Aleatory uncertainty: probabilities are considered as objective quantities and interpreted as frequencies or limits of frequencies
  - Epistemic uncertainty: probabilities are subjective, interpreted as degrees of belief
- Main objections against the use of probability theory as a model epistemic uncertainty (Bayesian model)
  1. Inability to represent ignorance
  2. Not a plausibility model of how people make decisions based on weak information

# The wine/water paradox

- Principle of Indifference (PI): in the absence of information about some quantity $X$, we should assign equal probability to any possible value of $X$
- The wine/water paradox

*There is a certain quantity of liquids. All that we know about the liquid is that it is composed entirely of wine and water, and the ratio of wine to water is between 1/3 and 3.*

*What is the probability that the ratio of wine to water is less than or equal to 2?*

# The wine/water paradox (continued)

- Let $X$ denote the ratio of wine to water. All we know is that $X \in [1/3, 3]$. According to the PI, $X \sim \mathcal{U}_{[1/3,3]}$. Consequently

$$P(X \leq 2) = (2 - 1/3)/(3 - 1/3) = 5/8$$

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- Now, let $Y = 1/X$ denote the ratio of water to wine. All we know is that $Y \in [1/3, 3]$. According to the PI, $Y \sim \mathcal{U}_{[1/3,3]}$. Consequently

$$P(Y \geq 1/2) = (3 - 1/2)/(3 - 1/3) = 15/16$$

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$$P(Y \geq 1/2) = (3 - 1/2)/(3 - 1/3) = 15/16$$

- However, $P(X \leq 2) = P(Y \geq 1/2)$!

# Ellsberg's paradox

- Suppose you have an urn containing 30 red balls and 60 balls, either black or yellow. You are given a choice between two gambles:
  - *A*: You receive 100 euros if you draw a red ball
  - *B*: You receive 100 euros if you draw a black ball

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  - *A*: You receive 100 euros if you draw a red ball
  - *B*: You receive 100 euros if you draw a black ball
- Also, you are given a choice between these two gambles (about a different draw from the same urn):
  - *C*: You receive 100 euros if you draw a red or yellow ball
  - *D*: You receive 100 euros if you draw a black or yellow ball

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  - *A*: You receive 100 euros if you draw a red ball
  - *B*: You receive 100 euros if you draw a black ball
- Also, you are given a choice between these two gambles (about a different draw from the same urn):
  - *C*: You receive 100 euros if you draw a red or yellow ball
  - *D*: You receive 100 euros if you draw a black or yellow ball
- Most people strictly prefer *A* to *B*, hence $P(red) > P(black)$, but they strictly prefer *D* to *C*, hence $P(black) > P(red)$

# Set-membership approach

- Partial knowledge about some variable $X$ is described by a set $E$ of possible values for $X$ (constraint)
- Example:
  - Consider a system described by the equation

    $$y = f(x_1, \ldots, x_n; \theta)$$

    where $y$ is the output, $x_1, \ldots, x_n$ are the inputs and $\theta$ is a parameter
  - Knowing that $x_i \in [\underline{x}_i, \overline{x}_i]$, $i = 1, \ldots, n$ and $\theta \in [\underline{\theta}, \overline{\theta}]$, find a set $\mathbb{Y}$ surely containing $y$
- Advantage: computationally simpler than the probabilistic approach in many cases (interval analysis)
- Drawback: no way to express doubt, conservative approach

# Theory of belief functions

## History

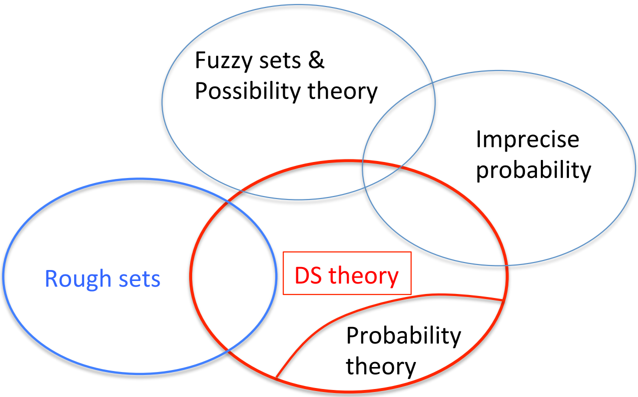
- A formal framework for representing and reasoning with uncertain information
- Also known as Dempster-Shafer theory or Evidence theory
- Originates from the work of Dempster (1968) in the context of statistical inference.
- Formalized by Shafer (1976) as a theory of evidence
- Popularized and developed by Smets in the 1980's and 1990's under the name Transferable Belief Model
- Starting from the 1990's, growing number of applications in information fusion, classification, reliability and risk analysis, etc.

# Theory of belief functions

## Main idea

- The theory of belief functions extends both the set-membership approach and Probability Theory
  - A belief function may be viewed both as a generalized set and as a non additive measure
  - The theory includes extensions of probabilistic notions (conditioning, marginalization) and set-theoretic notions (intersection, union, inclusion, etc.)
- Dempter-Shafer reasoning produces the same results as probabilistic reasoning or interval analysis when provided with the same information
- However, the greater expressive power of the theory of belief functions allows us to represent what we know in a more faithful way

# Relationships wth other theories



# Outline

1. Basic notions
   - Mass functions
   - Belief and plausibility functions
   - Dempster's rule

2. Selected advanced topics
   - Informational orderings
   - Cautious rule
   - Belief functions on product spaces
   - Belief functions on infinite spaces

# Outline

# Mass function

## Definition

- Let $X$ be a variable taking values in a finite set $\Omega$ (frame of discernment)
- Evidence about $X$ may be represented by a mass function $m : 2^\Omega \to [0, 1]$ such that

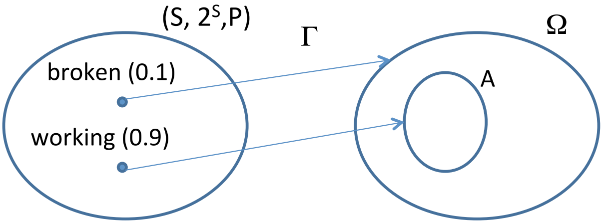
$$\sum_{A \subseteq \Omega} m(A) = 1$$

- Every $A$ of $\Omega$ such that $m(A) > 0$ is a focal set of $m$
- $m$ is said to be normalized if $m(\emptyset) = 0$. This property will be assumed hereafter, unless otherwise specified

# Example: the broken sensor

- Let $X$ be some physical quantity (e.g., a temperature), talking values in $\Omega$.
- A sensor returns a set of values $A \subset \Omega$, for instance, $A = [20, 22]$.
- However, the sensor may be broken, in which case the value it returns is completely arbitrary.
- There is a probability $p = 0.1$ that the sensor is broken.
- What can we say about $X$? How to represent the available information (evidence)?

# Analysis




- Here, the probability $p$ is not about $X$, but about the state of a sensor.
- Let $S = \{\text{working}, \text{broken}\}$ the set of possible sensor states.
  - If the state is "working", we know that $X \in A$.
  - If the state is "broken", we just know that $X \in \Omega$, and nothing more.
- This uncertain evidence can be represented by a mass function $m$ on $\Omega$, such that

$$m(A) = 0.9, \quad m(\Omega) = 0.1$$

# Source

- A mass function $m$ on $\Omega$ may be viewed as arising from
  - A set $S = \{s_1, \ldots, s_r\}$ of states (interpretations)
  - A probability measure $P$ on $S$
  - A multi-valued mapping $\Gamma : S \rightarrow 2^\Omega$
- The four-tuple $(S, 2^S, P, \Gamma)$ is called a source for $m$
- Meaning: under interpretation $s_i$, the evidence tells us that $X \in \Gamma(s_i)$, and nothing more. The probability $P(\{s_i\})$ is transferred to $A_i = \Gamma(s_i)$
- $m(A)$ is the probability of knowing that $X \in A$, and nothing more, given the available evidence

# Special cases

- If the evidence tells us that $X \in A$ for sure and nothing more, for some $A \subseteq \Omega$, then we have a logical mass function $m_{[A]}$ such that $m_{[A]}(A) = 1$
  - $m_{[A]}$ is equivalent to $A$
  - Special case: $m_?$, the vacuous mass function, represents total ignorance
- If each interpretation $s_i$ of the evidence points to a single value of $X$, then all focal sets are singletons and $m$ is said to be Bayesian. It is equivalent to a probability distribution
- A Dempster-Shafer mass function can thus be seen as
  - a generalized set
  - a generalized probability distribution
- Total ignorance is represented by the vacuous mass function $m_?$ such that $m_?(\Omega) = 1$

# Outline

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   - Dempster's rule

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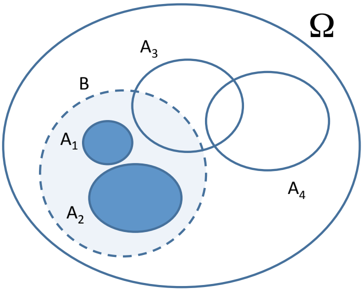
# Belief function

- If the evidence tells us that the truth is in $A$, and $A \subseteq B$, we say that the evidence supports $B$.



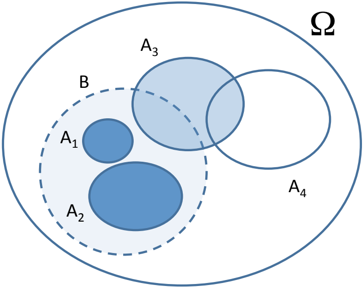

- Given a normalized mass function $m$, the probability that the evidence supports $B$ is thus

$$Bel(B) = \sum_{A \subseteq B} m(A)$$

- The number $Bel(B)$ is called the degree of belief in $B$, and the function $B \rightarrow Bel(B)$ is called a belief function.

# Plausibility function

- If the evidence does not support $\overline{B}$, it is consistent with $B$.




- The probability that the evidence is consistent with $B$ is thus

$$Pl(B) = \sum_{A \cap B \neq \emptyset} m(A)$$
$$= 1 - Bel(\overline{B}).$$

- The number $Pl(B)$ is called the plausibility of $B$, and the function $B \rightarrow Pl(B)$ is called a plausibility function.

# Two-dimensional representation

- The uncertainty on a proposition $B$ is represented by two numbers: $Bel(B)$ and $Pl(B)$, with $Bel(B) \leq Pl(B)$.
- The intervals $[Bel(B), Pl(B)]$ have maximum length when $m$ is the vacuous mass function. Then,

$$[Bel(B), Pl(B)] = [0, 1]$$

  for all subset $B$ of $\Omega$, except $\emptyset$ and $\Omega$.
- The intervals $[Bel(B), Pl(B)]$ are reduced to points when the focal sets of $m$ are singletons (m is then said to be Bayesian); then,

$$Bel(B) = Pl(B)$$

  for all $B$, and $Bel$ is a probability measure.

# Broken sensor example

- From

$$m(A) = 0.9, \quad m(\Omega) = 0.1$$

  we get

$$Bel(A) = m(A) = 0.9, \quad Pl(A) = m(A) + m(\Omega) = 1$$
$$Bel(\overline{A}) = 0, \quad Pl(\overline{A}) = m(\Omega) = 0.1$$
$$Bel(\Omega) = Pl(\Omega) = 1$$

- We observe that

$$Bel(A \cup \overline{A}) \geq Bel(A) + Bel(\overline{A})$$
$$Pl(A \cup \overline{A}) \leq Pl(A) + Pl(\overline{A})$$

- *Bel* and *Pl* are non additive measures.

# Characterization of belief functions

- Function $Bel : 2^\Omega \rightarrow [0,1]$ is a completely monotone capacity: it verifies $Bel(\emptyset) = 0$, $Bel(\Omega) = 1$ and

$$Bel\left(\bigcup_{i=1}^{k} A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} Bel\left(\bigcap_{i \in I} A_i\right).$$

  for any $k \geq 2$ and for any family $A_1, \ldots, A_k$ in $2^\Omega$.
- Conversely, to any completely monotone capacity $Bel$ corresponds a unique mass function $m$ such that:

$$m(A) = \sum_{\emptyset \neq B \subseteq A} (-1)^{|A|-|B|} Bel(B), \quad \forall A \subseteq \Omega.$$

# Relations between *m*, *Bel* and *Pl*

- Let $m$ be a mass function, $Bel$ and $Pl$ the corresponding belief and plausibility functions
- For all $A \subseteq \Omega$,

$$Bel(A) = 1 - Pl(\overline{A})$$

$$m(A) = \sum_{\emptyset \neq B \subseteq A} (-1)^{|A|-|B|} Bel(B)$$

$$m(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|+1} Pl(\overline{B})$$

- $m$, $Bel$ and $Pl$ are thus three equivalent representations of
  - a piece of evidence or, equivalently
  - a state of belief induced by this evidence

# Relationship with Possibility theory

- When the focal sets of $m$ are nested: $A_1 \subset A_2 \subset \ldots \subset A_r$, $m$ is said to be consonant
- The following relations then hold

$$Pl(A \cup B) = \max\left(Pl(A), Pl(B)\right), \quad \forall A, B \subseteq \Omega$$

- $Pl$ is this a possibility measure, and $Bel$ is the dual necessity measure
- The possibility distribution is the contour function

$$pl(x) = Pl(\{x\}), \quad \forall x \in \Omega$$

- The theory of belief function can thus be considered as more expressive than possibility theory (but the combination operations are different, see later).

# Credal set

- A probability measure $P$ on $\Omega$ is said to be compatible with $m$ if

$$\forall A \subseteq \Omega, \quad Bel(A) \leq P(A) \leq Pl(A)$$

- The set $\mathcal{P}(m)$ of probability measures compatible with $m$ is called the credal set of $m$

$$\mathcal{P}(m) = \{P : \forall A \subseteq \Omega, Bel(A) \leq P(A)\}$$

- $Bel$ is the lower envelope of $\mathcal{P}(m)$

$$\forall A \subseteq \Omega, \quad Bel(A) = \min_{P \in \mathcal{P}(m)} P(A)$$

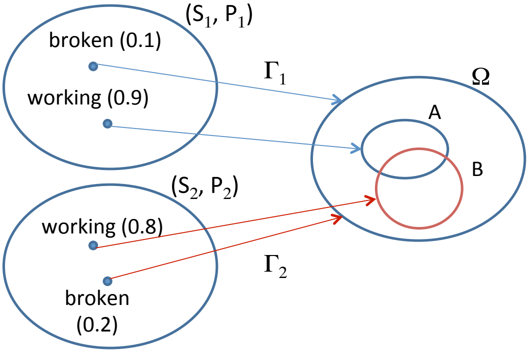
- Not all lower envelopes of sets of probability measures are belief functions!

# Outline

# Broken sensor example continued

- The first item of evidence gave us: $m_1(A) = 0.9$, $m_1(\Omega) = 0.1$.
- Another sensor returns another set of values $B$, and it is in working condition with probability 0.8.
- This second piece if evidence can be represented by the mass function: $m_2(B) = 0.8$, $m_2(\Omega) = 0.2$
- How to combine these two pieces of evidence?

# Analysis




- If interpretations $s_1 \in S_1$ and $s_2 \in S_2$ both hold, then $X \in \Gamma_1(s_1) \cap \Gamma_2(s_2)$
- If the two pieces of evidence are independent, then the probability that $s_1$ and $s_2$ both hold is $P_1(\{s_1\})P_2(\{s_2\})$

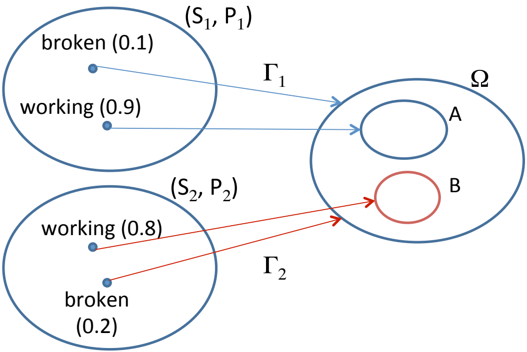
# Computation

| | $S_2$ working (0.8) | $S_2$ broken (0.2) |
|---|---|---|
| $S_1$ working (0.9) | $A \cap B$, 0.72 | $A$, 0.18 |
| $S_1$ broken (0.1) | $B$, 0.08 | $\Omega$, 0.02 |

We then get the following combined mass function,

$$m(A \cap B) = 0.72$$
$$m(A) = 0.18$$
$$m(B) = 0.08$$
$$m(\Omega) = 0.02$$

# Case of conflicting pieces of evidence




- If $\Gamma_1(s_1) \cap \Gamma_2(s_2) = \emptyset$, we know that $s_1$ and $s_2$ cannot hold simultaneously
- The joint probability distribution on $S_1 \times S_2$ must be conditioned to eliminate impossible pairs of interpretation

# Computation

| | $S_2$ working (0.8) | $S_2$ broken (0.2) |
|---|---|---|
| $S_1$ working (0.9) | $\emptyset$, 0.72 | $A$, 0.18 |
| $S_1$ broken (0.1) | $B$, 0.08 | $\Omega$, 0.02 |

We then get the following combined mass function,

$$\begin{aligned} m(\emptyset) &= 0 \\ m(A) &= 0.18/0.28 \approx 0.64 \\ m(B) &= 0.08/0.28 \approx 0.29 \\ m(\Omega) &= 0.02/0.28 \approx 0.07 \end{aligned}$$

# Dempster's rule

- Let $m_1$ and $m_2$ be two mass functions and

$$\kappa = \sum_{B \cap C = \emptyset} m_1(B) m_2(C)$$

their degree of conflict

- If $\kappa < 1$, then $m_1$ and $m_2$ can be combined as

$$(m_1 \oplus m_2)(A) = \frac{1}{1-\kappa} \sum_{B \cap C = A} m_1(B) m_2(C), \quad \forall A \neq \emptyset$$

and $(m_1 \oplus m_2)(\emptyset) = 0$

# Dempster's rule

## Properties

- Commutativity, associativity. Neutral element: $m_?$
- Generalization of intersection: if $m_{[A]}$ and $m_{[B]}$ are logical mass functions and $A \cap B \neq \emptyset$, then

$$m_{[A]} \oplus m_{[B]} = m_{[A \cap B]}$$

- If either $m_1$ or $m_2$ is Bayesian, then so is $m_1 \oplus m_2$ (as the intersection of a singleton with another subset is either a singleton, or the empty set).

# Dempster's conditioning

- Conditioning is a special case, where a mass function $m$ is combined with a logical mass function $m_{[A]}$. Notation:

$$m \oplus m_{[A]} = m(\cdot|A)$$

- It can be shown that

$$Pl(B|A) = \frac{Pl(A \cap B)}{Pl(A)}.$$

- Generalization of Bayes' conditioning: if $m$ is a Bayesian mass function and $m_{[A]}$ is a logical mass function, then $m \oplus m_{[A]}$ is a Bayesian mass function corresponding to the conditioning of $m$ by $A$

# Commonality function

- Commonality function: let $Q : 2^\Omega \rightarrow [0,1]$ be defined as

$$Q(A) = \sum_{B \supseteq A} m(B), \quad \forall A \subseteq \Omega$$

- Conversely,

$$m(A) = \sum_{B \supseteq A} (-1)^{|B \setminus A|} Q(B)$$

- $Q$ is another equivalent representation of a belief function.

# Commonality function and Dempster's rule

- Let $Q_1$ and $Q_2$ be the commonality functions associated to $m_1$ and $m_2$.
- Let $Q_1 \oplus Q_2$ be the commonality function associated to $m_1 \oplus m_2$.
- We have

$$(Q_1 \oplus Q_2)(A) = \frac{1}{1-\kappa} Q_1(A) \cdot Q_2(A), \quad \forall A \subseteq \Omega, A \neq \emptyset$$

$$(Q_1 \oplus Q_2)(\emptyset) = 1$$

- In particular, $pl(\omega) = Q(\{\omega\})$. Consequently,

$$pl_1 \oplus pl_2 = (1-\kappa)^{-1} pl_1 pl_2.$$

# Remarks on normalization

- Mass functions expressing pieces of evidence are always normalized
- Smets introduced the unnormalized Dempster's rule (TBM conjunctive rule $\cap\!\!\!\!\bigcirc$), which may yield an unnormalized mass function
- He proposed to interpret $m(\emptyset)$ as the mass committed to the hypothesis that $X$ might not take its value in $\Omega$ (open-world assumption)
- I now think that this interpretation is problematic, as $m(\emptyset)$ increases mechanically when combining more and more items of evidence
- Claim: unnormalized mass functions (and $\cap\!\!\!\!\bigcirc$) are convenient mathematically, but only normalized mass functions make sense
- In particular, *Bel* and *Pl* should always be computed from normalized mass functions

# TBM disjunctive rule

- Let $(S_1, P_1, \Gamma_1)$ and $(S_2, P_2, \Gamma_2)$ be sources associated to two pieces of evidence
- If interpretation $s_k \in S_k$ holds and piece of evidence $k$ is reliable, then we can conclude that $X \in \Gamma_k(s_k)$
- If interpretation $s \in S_1$ and $s_2 \in S_2$ both hold and we assume that at least one of the two pieces of evidence is reliable, then we can conclude that $X \in \Gamma_1(s_1) \cup \Gamma_2(s_2)$
- This leads to the TBM disjunctive rule:

$$(m_1 \textcircled{\cup} m_2)(A) = \sum_{B \cup C = A} m_1(B) m_2(C), \quad \forall A \subseteq \Omega$$

- $Bel_1 \textcircled{\cup} Bel_2 = Bel_1 \cdot Bel_2$

# Outline

# Informational comparison of belief functions

- Let $m_1$ and $m_2$ be two mass functions on $\Omega$
- In what sense can we say that $m_1$ is more informative (committed) than $m_2$?
- Special case:
  - Let $m_{[A]}$ and $m_{[B]}$ be two logical mass functions
  - $m_{[A]}$ is more committed than $m_{[B]}$ iff $A \subseteq B$
- Extension to arbitrary mass functions?

# Plausibility ordering

- $m_1$ is pl-more committed than $m_2$ (noted $m_1 \sqsubseteq_{pl} m_2$) if

$$Pl_1(A) \leq Pl_2(A), \quad \forall A \subseteq \Omega$$

  or, equivalently,

$$Bel_1(A) \geq Bel_2(A), \quad \forall A \subseteq \Omega$$

- Imprecise probability interpretation:

$$m_1 \sqsubseteq_{pl} m_2 \Leftrightarrow \mathcal{P}(m_1) \subseteq \mathcal{P}(m_2)$$

- Properties:
  - Extension of set inclusion:

$$m_{[A]} \sqsubseteq_{pl} m_{[B]} \Leftrightarrow A \subseteq B$$

  - Greatest element: vacuous mass function $m_?$

# Commonality ordering

- If $m_1 = m \oplus m_2$ for some $m$, and if there is no conflict between $m$ and $m_2$, then $Q_1(A) = Q(A)Q_2(A) \leq Q_2(A)$ for all $A \subseteq \Omega$
- This property suggests that smaller values of the commonality function are associated with richer information content of the mass function
- $m_1$ is q-more committed than $m_2$ (noted $m_1 \sqsubseteq_q m_2$) if

$$Q_1(A) \leq Q_2(A), \quad \forall A \subseteq \Omega$$

- Properties:
  - Extension of set inclusion:

$$m_{[A]} \sqsubseteq_q m_{[B]} \Leftrightarrow A \subseteq B$$

  - Greatest element: vacuous mass function $m_?$

# Strong (specialization) ordering

- $m_1$ is a specialization of $m_2$ (noted $m_1 \sqsubseteq_s m_2$) if $m_1$ can be obtained from $m_2$ by distributing each mass $m_2(B)$ to subsets of $B$:

$$m_1(A) = \sum_{B \subseteq \Omega} S(A,B) m_2(B), \quad \forall A \subseteq \Omega,$$

  where $S(A,B) =$ proportion of $m_2(B)$ transferred to $A \subseteq B$
- $S$: specialization matrix
- Properties:
  - Extension of set inclusion
  - Greatest element: $m_?$
  - $m_1 \sqsubseteq_s m_2 \Rightarrow \begin{cases} m_1 \sqsubseteq_{pl} m_2 \\ m_1 \sqsubseteq_q m_2 \end{cases}$

# Least Commitment Principle

## Definition

**Definition (Least Commitment Principle)**

*When several belief functions are compatible with a set of constraints, the least informative according to some informational ordering (if it exists) should be selected*

A very powerful method for constructing belief functions!

# Outline

# Cautious rule

## Motivations

- The basic rules $\oplus$ and $\bigcirc\!\!\!\!\cup$ assume the sources of information to be independent, e.g.
  - experts with non overlapping experience/knowledge
  - non overlapping datasets
- What to do in case of non independent evidence?
  - Describe the nature of the interaction between sources (difficult, requires a lot of information)
  - Use a combination rule that tolerates redundancy in the combined information
- Such rules can be derived from the LCP using suitable informational orderings

# Cautious rule

## Principle

- Two sources provide mass functions $m_1$ and $m_2$, and the sources are both considered to be reliable
- After receiving these $m_1$ and $m_2$, the agent's state of belief should be represented by a mass function $m_{12}$ more committed than $m_1$, and more committed than $m_2$
- Let $\mathcal{S}_x(m)$ be the set of mass functions $m'$ such that $m' \sqsubseteq_x m$, for some $x \in \{pl, q, s, \cdots\}$. We thus impose that

$$m_{12} \in \mathcal{S}_x(m_1) \cap \mathcal{S}_x(m_2)$$

- According to the LCP, we should select the $x$-least committed element in $\mathcal{S}_x(m_1) \cap \mathcal{S}_x(m_2)$, if it exists

# Cautious rule

## Problem

- The above approach works for special cases
- Example (Dubois, Prade, Smets 2001): if $m_1$ and $m_2$ are consonant, then the $q$-least committed element in $\mathcal{S}_q(m_1) \cap \mathcal{S}_q(m_2)$ exists and it is unique: it is the consonant mass function with commonality function $Q_{12} = \min(Q_1, Q_2)$
- In general, neither existence nor uniqueness of a solution can be guaranteed with any of the $x$-orderings, $x \in \{pl, q, s\}$
- We need to define a new ordering relation

# Simple and separable mass functions

- Definition: $m$ is simple mass function if it has the following form

$$\begin{aligned} m(A) &= 1 - w(A) \\ m(\Omega) &= w(A) \end{aligned}$$

  for some $A \subset \Omega$, $A \neq \emptyset$ and $w(A) \in [0, 1]$. It is denoted by $A^{w(A)}$.
- Property: $A^{w_1(A)} \oplus A^{w_2(A)} = A^{w_1(A) w_2(A)}$
- A (normalized) mass function is separable if it can be written as the $\oplus$ combination of simple mass functions

$$m = \bigoplus_{\emptyset \neq A \subset \Omega} A^{w(A)}$$

  with $0 \leq w(A) \leq 1$ for all $A \subset \Omega$, $A \neq \emptyset$

# The $w$-ordering

- Let $m_1$ and $m_2$ be two mass functions
- We say that $m_1$ is $w$-less committed than $m_2$ (denoted by $m_1 \sqsubseteq_w m_2$) if

$$m_1 = m_2 \oplus m$$

  for some separable mass function $m$
- How to check this condition?

# Weight function

## Definition

- Let $m$ be a non dogmatic mass function, i.e., $m(\Omega) > 0$
- The weight function $w : 2^\Omega \rightarrow (0, +\infty)$ is defined by $w(\Omega) = 1$ and

$$\ln w(A) = -\sum_{B \supseteq A} (-1)^{|B|-|A|} \ln Q(B), \quad \forall A \subset \Omega$$

- It can be shown that $Q$ can be recovered from $w$ as follows

$$\ln Q(A) = -\sum_{\Omega \supset B \not\supseteq A} \ln w(B), \quad \forall A \subseteq \Omega$$

- $m$ can also be recovered from $w$ by

$$m = \bigoplus_{\emptyset \neq A \subset \Omega} A^{w(A)}$$

although $A^{w(A)}$ is not a proper mass function when $w(A) > 1$

# Weight function

## Properties

- $m$ is separable iff

$$w(A) \leq 1, \quad \forall A \subset \Omega, A \neq \emptyset$$

- Dempster's rule can be computed using the $w$-function by

$$m_1 \oplus m_2 = \bigoplus_{\emptyset \neq A \subset \Omega} A^{w_1(A) w_2(A)}$$

- Characterization of the $w$-ordering

$$m_1 \sqsubseteq_w m_2 \Leftrightarrow w_1(A) \leq w_2(A), \quad \forall A \subset \Omega, A \neq \emptyset$$

# Cautious rule

## Definition

- Let $m_1$ and $m_2$ be two non dogmatic mass functions with weight functions $w_1$ and $w_2$
- The $w$-least committed element in $\mathcal{S}_w(m_1) \cap \mathcal{S}_w(m_2)$ exists and is unique. It is defined by:

$$m_1 \textcircled{\wedge} m_2 = \bigoplus_{\emptyset \neq A \subset \Omega} A^{\min(w_1(A), w_2(A))}$$

- Operator $\textcircled{\wedge}$ is called the (normalized) cautious rule

# Computation

## Cautious rule computation

| $m$-space | | $w$-space |
|---|---|---|
| $m_1$ | $\longrightarrow$ | $w_1$ |
| $m_2$ | $\longrightarrow$ | $w_2$ |
| $m_1 \textcircled{\wedge} m_2$ | $\longleftarrow$ | $\min(w_1, w_2)$ |

Remark: we often have simple mass functions in the first place, so that the $w$ function is readily available.

# Cautious rule

## Properties

- Commutative, associative
- Idempotent : $\forall m,\ m \owedge m = m$
- Distributivity of $\oplus$ with respect to $\owedge$

$$(m_1 \oplus m_2) \owedge (m_1 \oplus m_3) = m_1 \oplus (m_2 \owedge m_3), \forall m_1, m_2, m_3$$

The common item of evidence $m_1$ is not counted twice!

- No neutral element, but $m_? \owedge m = m$ iff $m$ is separable

# Basic rules

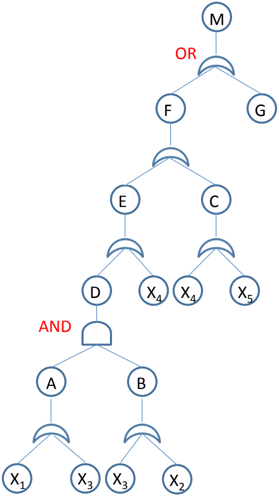
| Sources | independent | dependent |
|---|---|---|
| All reliable | $\oplus$ | $\owedge$ |
| At least one reliable | $\ovee$ (∪) | $\ovee$ |

$\ovee$ is the bold disjunctive rule

# Outline

# Belief functions on product spaces

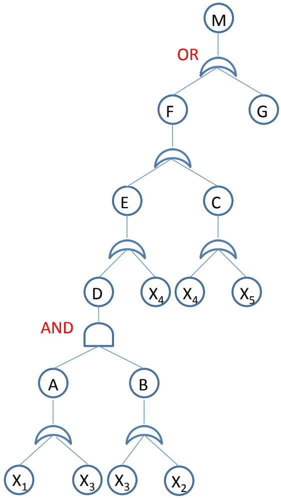
## Motivation

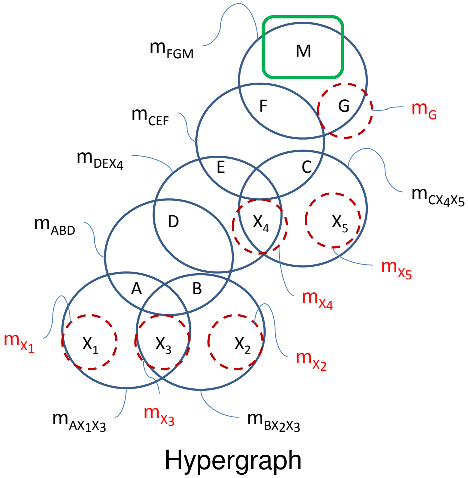



- In many applications, we need to express uncertain information about several variables taking values in different domains
- Example: fault tree (logical relations between Boolean variables and probabilistic or evidential information about elementary events)

# Fault tree example

(Dempster & Kong, 1988)



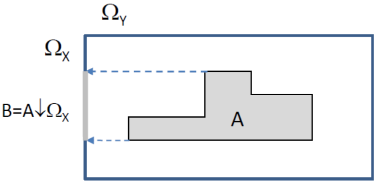



# Multidimensional belief functions

## Marginalization, vacuous extension

- Let $X$ and $Y$ be two variables defined on frames $\Omega_X$ and $\Omega_Y$
- Let $\Omega_{XY} = \Omega_X \times \Omega_Y$ be the product frame
- A mass function $m_{XY}$ on $\Omega_{XY}$ can be seen as an generalized relation between variables $X$ and $Y$
- Two basic operations on product frames
  1. Express a joint mass function $m_{XY}$ in the coarser frame $\Omega_X$ or $\Omega_Y$ (marginalization)
  2. Express a marginal mass function $m_X$ on $\Omega_X$ in the finer frame $\Omega_{XY}$ (vacuous extension)

# Marginalization



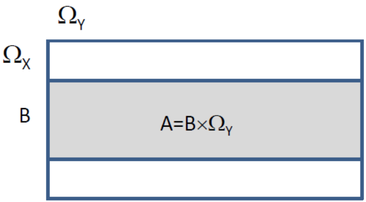

- Problem: express $m_{XY}$ in $\Omega_X$
- Solution: transfer each mass $m_{XY}(A)$ to the projection of $A$ on $\Omega_X$

- Marginal mass function

$$m_{XY\downarrow X}(B) = \sum_{\{A\subseteq\Omega_{XY}, A\downarrow\Omega_X=B\}} m_{XY}(A) \quad \forall B \subseteq \Omega_X$$

- Generalizes both set projection and probabilistic marginalization

# Vacuous extension




- Problem: express $m_X$ in $\Omega_{XY}$
- Solution: transfer each mass $m_X(B)$ to the cylindrical extension of $B$: $B \times \Omega_Y$

- Vacuous extension:

$$m_{X\uparrow XY}(A) = \begin{cases} m_X(B) & \text{if } A = B \times \Omega_Y \\ 0 & \text{otherwise} \end{cases}$$

# Operations in product frames

## Application to approximate reasoning

- Assume that we have:
  - Partial knowledge of $X$ formalized as a mass function $m_X$
  - A joint mass function $m_{XY}$ representing an uncertain relation between $X$ and $Y$
- What can we say about $Y$?
- Solution:

$$m_Y = \left(m_{X\uparrow XY} \oplus m_{XY}\right)_{\downarrow Y}$$

- Simpler notation:

$$m_Y = \left(m_X \oplus m_{XY}\right)_{\downarrow Y}$$

- Infeasible with many variables and large frames of discernment, but efficient algorithms exist to carry out the operations in frames of minimal dimensions

# Outline

1. Basic notions
   - Mass functions
   - Belief and plausibility functions
   - Dempster's rule
2. Selected advanced topics
   - Informational orderings
   - Cautious rule
   - Belief functions on product spaces
   - Belief functions on infinite spaces

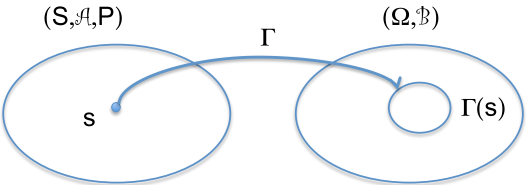
# Belief function: general definition

- Let $\Omega$ be a set (finite or not) and $\mathcal{B}$ be an algebra of subsets of $\Omega$ (a nonempty family of subsets of $\Omega$, closed under complementation and finite intersection).
- A belief function (BF) on $\mathcal{B}$ is a mapping $Bel : \mathcal{B} \rightarrow [0, 1]$ verifying $Bel(\emptyset) = 0$, $Bel(\Omega) = 1$ and the complete monotonicity property: for any $k \geq 2$ and any collection $B_1, \ldots, B_k$ of elements of $\mathcal{B}$,

$$Bel\left(\bigcup_{i=1}^{k} B_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} Bel\left(\bigcap_{i \in I} B_i\right)$$

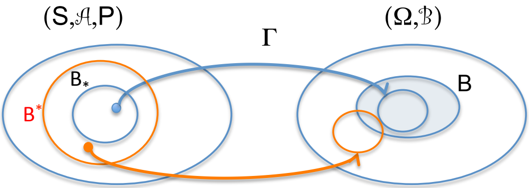
- A function $Pl : \mathcal{B} \rightarrow [0, 1]$ is a plausibility function iff $B \rightarrow 1 - Pl(\overline{B})$ is a belief function

# Source




- Let $S$ be a state space, $\mathcal{A}$ an algebra of subsets of $S$, $\mathbb{P}$ a finitely additive probability on $(S, \mathcal{A})$
- Let $\Omega$ be a set and $\mathcal{B}$ an algebra of subsets of $\Omega$
- $\Gamma$ a multivalued mapping from $S$ to $2^{\Omega} \setminus \{\emptyset\}$
- The four-tuple $(S, \mathcal{A}, \mathbb{P}, \Gamma)$ is called a source
- Under some conditions, it induces a belief function on $(\Omega, \mathcal{B})$

# Strong measurability




- Lower and upper inverses: for all $B \in \mathcal{B}$,

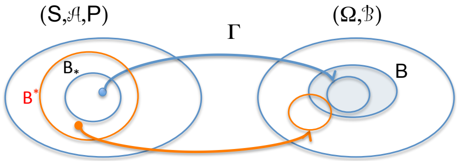
$$\Gamma_*(B) = B_* = \{s \in S | \Gamma(s) \neq \emptyset, \Gamma(s) \subseteq B\}$$

$$\Gamma^*(B) = B^* = \{s \in S | \Gamma(s) \cap B \neq \emptyset\}$$

- $\Gamma$ is strongly measurable wrt $\mathcal{A}$ and $\mathcal{B}$ if, for all $B \in \mathcal{B}$, $B^* \in \mathcal{A}$
- $(\forall B \in \mathcal{B}, B^* \in \mathcal{A}) \Leftrightarrow (\forall B \in \mathcal{B}, B_* \in \mathcal{A})$

# Belief function induced by a source

## Lower and upper probabilities

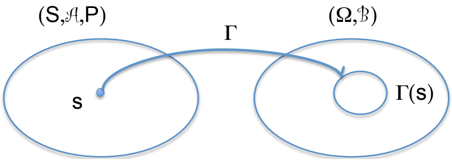



- Lower and upper probabilities:

$$\forall B \in \mathcal{B}, \quad \mathbb{P}_*(B) = \frac{\mathbb{P}(B_*)}{\mathbb{P}(\Omega^*)}, \quad \mathbb{P}^*(B) = \frac{\mathbb{P}(B^*)}{\mathbb{P}(\Omega^*)} = 1 - Bel(\overline{B})$$

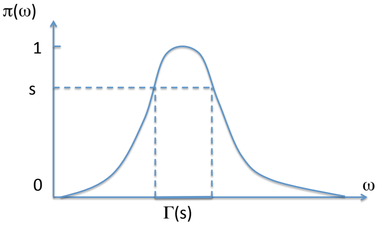
- $\mathbb{P}_*$ is a BF, and $\mathbb{P}^*$ is the dual plausibility function
- Conversely, for any belief function, there is a source that induces it (Shafer's thesis, 1973)

# Interpretation




- Typically, $\Omega$ is the domain of an unknown quantity $\omega$, and $S$ is a set of interpretations of a given piece of evidence about $\omega$
- If $s \in S$ holds, then the evidence tells us that $\omega \in \Gamma(s)$, and nothing more
- Then
  - $Bel(B)$ is the probability that the evidence supports $B$
  - $Pl(B)$ is the probability that the evidence is consistent with $B$

# Consonant belief function



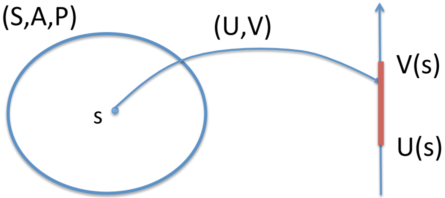

- Let $\pi$ be a mapping from $\Omega$ to $S = [0, 1]$ s.t. $\sup \pi = 1$
- Let $\Gamma$ be the multi-valued mapping from $S$ to $2^\Omega$ defined by

$$\forall s \in [0, 1], \quad \Gamma(s) = \{\omega \in \Omega | \pi(\omega) \geq s\}$$

- The source $(S, \mathcal{B}(S), \lambda, \Gamma)$ defines a consonant BF on $\Omega$, such that $pl(\omega) = \pi(\omega)$ (contour function)
- The corresponding plausibility function is a possibility measure

$$\forall B \subseteq \Omega, \quad Pl(B) = \sup_{\omega \in B} pl(\omega)$$

# Random closed interval

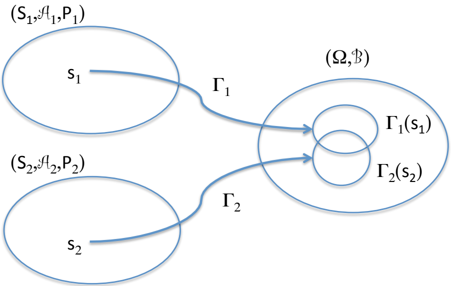



- Let $(U, V)$ be a bi-dimensional random vector from a probability space $(S, \mathcal{A}, \mathbb{P})$ to $\mathbb{R}^2$ such that $U \leq V$ a.s.
- Multi-valued mapping:

$$\Gamma : s \rightarrow \Gamma(s) = [U(s), V(s)]$$

- The source $(S, \mathcal{A}, \mathbb{P}, \Gamma)$ is a random closed interval. It defines a BF on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

# Dempster's rule




- Let $(S_i, \mathcal{A}_i, \mathbb{P}_i, \Gamma_i)$, $i = 1, 2$ be two sources representing independent items of evidence, inducing BF $Bel_1$ and $Bel_2$
- The combined BF $Bel = Bel_1 \oplus Bel_2$ is induced by the source $(S_1 \times S_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{P}_1 \otimes \mathbb{P}_2, \Gamma_\cap)$ with

$$\Gamma_\cap(s_1, s_2) = \Gamma_1(s_1) \cap \Gamma_2(s_2)$$

# Approximate computation

## Monte Carlo simulation

**Require:** Desired number of focal sets $N$

$i \leftarrow 0$

**while** $i < N$ **do**

Draw $s_1$ in $S_1$ from $\mathbb{P}_1$

Draw $s_2$ in $S_2$ from $\mathbb{P}_2$

$\Gamma_\cap(s_1, s_2) \leftarrow \Gamma_1(s_1) \cap \Gamma_2(s_2)$

**if** $\Gamma_\cap(s_1, s_2) \neq \emptyset$ **then**

$i \leftarrow i + 1$

$B_i \leftarrow \Gamma_\cap(s_1, s_2)$

**end if**

**end while**

$\widehat{Bel}(B) \leftarrow \frac{1}{N} \#\{i \in \{1, \ldots, N\} | B_i \subseteq B\}$

$\widehat{Pl}(B) \leftarrow \frac{1}{N} \#\{i \in \{1, \ldots, N\} | B_i \cap B \neq \emptyset\}$

# Summary

- The theory of belief functions: a very general formalism for representing imprecision and uncertainty that extends both probabilistic and set-theoretic frameworks
  - Belief functions can be seen both as generalized sets and as generalized probability measures
  - Reasoning mechanisms extend both set-theoretic notions (intersection, union, cylindrical extension, inclusion relations, etc.) and probabilistic notions (conditioning, marginalization, Bayes theorem, stochastic ordering, etc.)
- The theory of belief function can also be seen as more geneal than Possibility theory (possibility measures are particular plausibility functions)
- The mathematical theory of belief functions in infinite spaces exists. We need practical models

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